Supplementary Section 6S.10

Alternative Notations

Just as we can express the same thoughts in different languages, 'He has a big head' and 'El tiene una cabeza grande', there are many different ways to express the same logical claims. Some of these differences are thinly cosmetic. Others are more interesting.

Insofar as the different systems of notation we'll examine in this section are merely different ways of expressing the same logic, they are not particularly important. But one of the most frustrating aspects of studying logic, at first, is getting comfortable with different systems of notation. So it's good to try to get comfortable with a variety of different ways of presenting logic.

Most simply, there are different symbols for all of the logical operators. You can easily find some by perusing various logical texts and websites. The following table contains the most common.

Operator	We use	Others use		
Negation	$\sim P$	$\neg P$	$-P$	$\overline{P}$
Conjunction	$P \bullet Q$	$P \wedge Q$	$P \& Q$	PQ
Disjunction	$P \vee Q$	$P + Q$		
Material conditional	$P \supset Q$	$P \rightarrow Q$	$P \Rightarrow Q$	
Biconditional	$P \equiv Q$	$P \leftrightarrow Q$	$P \Leftrightarrow Q$	$P \sim Q$
Existential quantifier	$\exists$	Σ	$\vee$	
Universal quantifier	$\forall$	Π	$\wedge$	

There are also propositional operators that do not appear in our logical system at all. For example, there are two unary operators called the Sheffer stroke (|) and the Peirce arrow (↓). With these operators, we can define all five of the operators of **PL**. Such operators may be used for systems in which one wants a minimal vocabulary and in which one does not need to have simplicity of expression. The balance between simplicity of vocabulary and simplicity of expression is a deep topic, but not one we'll engage in this section.

Another example of a logical symbol that has no equivalent in our system is one used for non-material conditionals; often just an arrow is used. Some systems of logic have symbols for truth and falsity within the object language: top ($\top$) and bottom ($\bot$); our 1s and 0s are metalinguistic, appearing in the truth tables, but not in the vocabulary of the object language.

One drawback of the languages in this book is that we have limited numbers of terms: only twenty-six propositional variables in **PL**, only five quantifier and singular-term variables in **M**, and so on. One easy and common way of formulating a logical language with indefinitely many terms is to allow a function like $'$ or $*$ to distinguish different terms.

P, P', P'', P''', P''''
x, x^*, x^{**}, x^{***}

Such notations are austere and fecund, but difficult to read. Our 6S.10.1, for example, looks like 6S.10.2 using the former option.

6S.10.1 $[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$
6S.10.2 $[P \supset (P' \supset P'')] \supset [(P \supset P') \supset (P \supset P'')]$

The more propositional variables one uses in any formula, the harder it is to discern the different ones from each other.

Another variant of our system of logic, perhaps not merely notational, allows propositional variables like the ones we used in **PL** in predicate logic. Such systems may use capital letters followed by no singular terms as zero-place predicates, like propositional variables.

FREGE'S ORIGINAL NOTATION

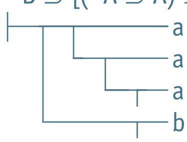
In *Begriffsschrift*, Frege systematically unified Aristotelian categorical logic and Stoic propositional logic, revolutionizing the field. But his system of notation is notoriously cumbersome. Not only did Frege divide inferences into separate lines, vertically, he also divided individual propositions into separate lines, according to their component parts. He also used lower-case letters for propositional variables. So where we write ~A, Frege wrote 6S.10.3.

6S.10.3 

And for our ‘A ⊃ B’, Frege wrote 6S.10.4.

6S.10.4 

Proposition 40 of *Begriffsschrift*, which we write as 6S.10.5, Frege writes as 6S.10.6.

6S.10.5 $\sim B \supset [(\sim A \supset A) \supset A]$
6S.10.6 

Things get uglier from there. Quantifiers are placed in little notched depressions in the horizontal lines of a formula. Functions are distinguished from variables typographically by varying fonts. It's all a little overwhelming to try to parse.

Fortunately, there are good presentations of Frege's work in modern notation. See the appendix to Richard Mendelsohn's book on Frege's work and George Boolos's article, both in the suggested readings list below.

POLISH NOTATION

One of the striking advantages of Frege's notation, despite its vertical complexities, is that there is no need for the kind of punctuation we use. All formulas of Frege's *Begriffsschrift* are unambiguous without any parentheses or brackets.

A system of notation developed in the early twentieth century by the great Polish logician Jan Łukasiewicz also avoids brackets, but uses a system more similar to ours. Polish notation, as it's now known, uses lower-case letters for propositional variables and upper-case letters for the logical operators.

Operator	We use	Polish
Negation	~P	Np
Conjunction	P • Q	Kpq
Disjunction (a.k.a. alternation)	P ∨ Q	Apq
Material conditional	P ⊃ Q	Cpq
Biconditional	P ≡ Q	Epq
Existential quantifier	∃	Σ
Universal quantifier	∀	Π

When combining truth functions, one always starts with the main operator and puts the whole subformula, including its main operator, in place of the single variable in the foregoing chart.

Our Notation	Polish Notation
P • ~Q	KpNq
~P ∨ Q	ANpq
~(P ⊃ ~Q)	NCpNq
~(P ≡ ~Q) • (~P ≡ Q)	KNEpNqENpq

Despite, or perhaps because of, the lack of punctuation, propositions written in Polish notation can be difficult to parse because all the terms tend to get packed together. Still, Polish notation has the great advantage of leading with the main operator. The first letter of any wff or sub-wff is the main operator of that proposition. This makes truth tables particularly perspicuous. Take, for example, the proposition 2.4.2, which I'll write in Polish notation as 6S.10.7.

2.4.2	$[(P \supset Q) \bullet (Q \supset R)] \supset (P \supset R)$
6S.10.7	CKCpqCqrCpr

You can turn back to chapter 2 for the truth table for 2.4.2 in our notation. In Polish notation, the truth tables are always easy to read, since the main operator is always in the front of the formula.

p	q	r	C	K	C	p	q	C	q	r	C	p	r
1	1	1	**1**	1	1	1	1	1	1	1	1	1	1
1	1	0	**1**	0	1	1	1	0	1	0	0	1	0
1	0	1	**1**	0	0	1	0	1	0	1	1	1	1
1	0	0	**1**	0	0	1	0	1	0	0	0	1	0
0	1	1	**1**	1	1	0	1	1	1	1	1	0	1
0	1	0	**1**	0	1	0	1	0	1	0	1	0	0
0	0	1	**1**	1	1	0	0	1	0	1	1	0	1
0	0	0	**1**	1	1	0	0	1	0	0	1	0	0

It's worth a moment, and is not unpleasant if you like this sort of thing, to convince yourself that the bunched-up formulas of Polish notation actually are unambiguous. There are exercises at the end of this section that you can use to practice translating between our notation for **PL** and Polish notation.

TELL ME MORE

- What are the Sheffer stroke (|) and the Peirce arrow (↓)? See 6S.8: Adequacy.

EXERCISES 6S.10a

Convert each formula from Polish notation to **PL**.

1. CNpq
2. KpNq
3. ApAqr
4. CpCqr
5. CCpqr
6. NEpCpq
7. CKpNqr
8. ApKNqNr
9. CKpNqAqr
10. CNEpqKrNs
11. CCCpCqpCCCNrCsNtCCrCsuCCtsCtuvCwv

EXERCISES 6S.10b

Convert each formula from **PL** to Polish notation.

1. ~P ∨ Q
2. M • ~A
3. P ≡ (D • G)
4. (F • L) • ~C
5. ~(M ∨ S)
6. (O • T) ≡ ~R
7. C ⊃ [J • (P • I)]
8. ~D ≡ (~P • ~T)
9. [S ⊃ (~P • ~C)] ∨ (R • ~D)
10. (~T ⊃ U) • (~V ⊃ W)
11. P ⊃ [~(Q ⊃ R) ⊃ (Q ∨ ~R)]

For Further Research and Writing

1. What are the advantages of Frege's *Begriffsschrift* notation and Polish notation over our horizontal notation? How might we change our system to accommodate those advantages?
2. W. V. Quine, in his influential *Method of Logic*, criticizes Łukasiewicz's Polish notation for being imperspicuous. In its place, he introduces dots that indicate the order of formulas. Quine's system of dots also avoids brackets and parentheses, and allows for series of conjunctions and disjunctions. Compare and contrast the perspicuity of the three systems: ours, Quine's, and Łukasiewicz's. What other factors might favor one system of notation over others?
3. Construct truth tables for some of the formulas from Exercises 6S.10a. What advantages and disadvantages do you find for working in Polish notation?

Suggested Readings

Boolos, George. "Reading the *Begriffsschrift*." *Mind* 94 (1985): 33–44. This article, along with the appendix to Mendelsohn's book, contains Frege's logic in a more contemporary notation.

Frege, Gottlob. *Begriffsschrift*. In *From Frege to Gödel*, edited by Jean van Heijenort, 1–82. Cambridge, MA: Harvard University Press, 1982.

Kneale, W., and M. Kneale. *The Development of Logic* Oxford, UK: Clarendon Press, 1962. Section IX.1 is a discussion of varieties of symbolism.

Mendelsohn, Richard. *The Philosophy of Gottlob Frege*. Cambridge: Cambridge University Press, 2005. The appendix, along with Boolos's article, contains Frege's logic in a more contemporary notation.

Quine, W. V. *Methods of Logic*, 4th ed. Cambridge, MA: Harvard University Press, 1982. See section 4.

SOLUTIONS TO EXERCISES 6S.10a

1. $\sim P \supset Q$
2. $P \bullet \sim Q$
3. $P \vee (Q \vee R)$
4. $P \supset (Q \supset R)$
5. $(P \supset Q) \supset R$
6. $\sim[P \equiv (P \supset Q)]$
7. $(P \bullet \sim Q) \supset R$
8. $P \vee (\sim Q \bullet \sim R)$
9. $(P \bullet \sim Q) \supset (Q \vee R)$
10. $\sim(P \equiv Q) \supset (R \bullet \sim S)$
11. $\{[P \supset (Q \supset P)] \supset \{[\sim R \supset (S \supset \sim T)] \supset \{[R \supset (S \supset U)] \supset [(T \supset S) \supset (T \supset U)]\}$ $\supset V\}\} \supset (W \supset V)$

SOLUTIONS TO EXERCISES 6S.10b

1. ANpq
2. KmNa
3. EpKdg
4. KKflNc
5. NAms
6. EKotNr
7. CcKjKpi
8. ENdKNpNt
9. ACsKNpNcKrNd
10. KCNtuCNvw
11. CpCNCqrAqNr