3 Trigonometry

(1) Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$, and express $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$.

$$\cos 2\theta = 2\cos^2\theta - 1 \quad \Rightarrow \quad \begin{aligned} \cos 4\theta &= 2\cos^2 2\theta - 1 \\ &= 2\left(2\cos^2\theta - 1\right)^2 - 1 \\ &= 2\left(4\cos^4\theta - 4\cos^2\theta + 1\right) - 1 \\ &= \underline{8\cos^4\theta - 8\cos^2\theta + 1} \end{aligned}$$

$$\sin 2\theta = 2\sin\theta\cos\theta \quad \Rightarrow \quad \begin{aligned} \sin 4\theta &= 2\sin 2\theta\cos 2\theta \\ &= 2\left(2\sin\theta\cos\theta\right)\left(\cos^2\theta - \sin^2\theta\right) \\ &= \underline{4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta} \end{aligned}$$

(2) Show that $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 4\cos\theta\cos 2\theta\cos 4\theta$.

$$\cos A + \cos B = 2\cos\left(\tfrac{A+B}{2}\right)\cos\left(\tfrac{A-B}{2}\right)$$

$\cos(-\theta) = \cos\theta$

$$\therefore \quad \cos\theta + \cos 3\theta = 2\cos 2\theta\cos\theta \quad \text{and} \quad \cos 5\theta + \cos 7\theta = 2\cos 6\theta\cos\theta$$

$$\therefore \quad \begin{aligned} \cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta &= 2\left(\cos 2\theta + \cos 6\theta\right)\cos\theta \\ &= 2\left(2\cos 4\theta\cos 2\theta\right)\cos\theta \\ &= \underline{4\cos\theta\cos 2\theta\cos 4\theta} \end{aligned}$$

(3) By using a factor formula, find the values of θ between 0 and π which satisfy the equation $\cos\theta = \cos 2\theta + \cos 4\theta$.

$$\cos 2\theta + \cos 4\theta = 2\cos 3\theta \cos\theta$$

$$\therefore \quad \cos\theta = \cos 2\theta + \cos 4\theta \quad \Rightarrow \quad \cos\theta\,(1 - 2\cos 3\theta) = 0$$

$$\therefore \quad \cos\theta = 0 \quad \text{or} \quad \cos 3\theta = \tfrac{1}{2}$$

$$\therefore \quad \theta = \tfrac{\pi}{2} \quad \text{or} \quad 3\theta = \tfrac{\pi}{3} \quad \text{or} \quad \tfrac{5\pi}{3} \quad \text{or} \quad \tfrac{7\pi}{3}$$

$$\text{i.e.} \quad \underline{\theta = \tfrac{\pi}{9} \quad \text{or} \quad \tfrac{\pi}{2} \quad \text{or} \quad \tfrac{5\pi}{9} \quad \text{or} \quad \tfrac{7\pi}{9}}$$