4 Differentiation

(1) Differentiate $y = 1/x^2$ from 'first principles'.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{1/(x+\delta x)^2 - 1/x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^2 - (x+\delta x)^2}{(x+\delta x)^2\, x^2\, \delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^2 - x^2 - 2x\,\delta x - \delta x^2}{(x+\delta x)^2\, x^2\, \delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2x - \delta x}{(x+\delta x)^2\, x^2} = \frac{-2x}{x^2\, x^2} = -\frac{2}{x^3}$$

$$\therefore \quad \underline{\frac{\mathrm{d}}{\mathrm{d}x}(x^{-2}) = -2\,x^{-3}}$$

(2) By first taking logarithms, differentiate $y = a^x$ where a is a constant.

$$y = a^x \quad \Rightarrow \quad \ln y = \ln(a^x) = x \ln a \quad \text{— (1)}$$

$$\therefore \quad \frac{\mathrm{d}}{\mathrm{d}x}(1) \quad \Rightarrow \quad \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln a$$

$$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} = y \ln a = a^x \ln a$$

$$\text{i.e.} \quad \underline{\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a}$$

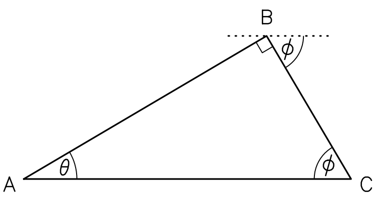
A simple check on this formula is provided by the special case of $a = e$, whence $\mathrm{d}/\mathrm{d}x(\mathrm{e}^x) = \mathrm{e}^x$ is recovered because $\ln e = 1$.

(3) Give an argument for why the gradient of a straight line perpendicular to $y = mx + c$ is $-1/m$.

$$\text{Slope of } \overrightarrow{AB} = \tan\theta = \frac{BC}{AB}$$

$$\text{Slope of } \overrightarrow{BC} = -\tan\phi = -\frac{AB}{BC}$$

$$\therefore \text{ Slope of } \overrightarrow{AB} = \frac{-1}{\text{Slope of } \overrightarrow{BC}}$$



i.e. Gradient of line perpendicular to $y = mx + c$ is $-1/m$.

There is an alternative more algebraic proof of this result, but it is somewhat longer than the geometric argument above. We let the coordinates of the point of intersection be (x_0, y_0), and those of two arbitrary points on the respective lines, with slopes m and μ, be (x_1, y_1) and (x_2, y_2). Then, from the definition of a gradient, and Pythagoras' theorem, we have

$$m = \frac{y_1 - y_0}{x_1 - x_0} \quad \text{and} \quad \mu = \frac{y_2 - y_0}{x_2 - x_0}, \quad \text{and}$$

$$\underbrace{(x_1 - x_0)^2 + (y_1 - y_0)^2}_{AB^2} + \underbrace{(x_2 - x_0)^2 + (y_2 - y_0)^2}_{BC^2} = \underbrace{(x_1 - x_2)^2 + (y_1 - y_2)^2}_{AC^2}$$

With a suitable expansion, and cancellation, of the last equation, it is not very difficult to show that the three relationships lead to the result $\mu = -1/m$.