5 Integration

(1) Given that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \qquad \text{and} \qquad \lim_{x\to\infty} x^n\, e^{-x^2} = 0$$

calculate

$$I_n = \int_0^\infty x^n\, e^{-\beta x^2}\,dx$$

where β is a positive constant and n is a non-negative integer.

Substituting $y = x\sqrt{\beta}$, so that $dy = dx\sqrt{\beta}$,

$$I_0 = \int_0^\infty e^{-\beta x^2}\,dx = \frac{1}{\sqrt{\beta}}\int_0^\infty e^{-y^2}\,dy = \frac{1}{2}\sqrt{\frac{\pi}{\beta}}$$

The $n=1$ case is straightforward because, to within a constant, the integrand is the derivative of $e^{-\beta x^2}$:

$$I_1 = \int_0^\infty x\, e^{-\beta x^2}\,dx = \left[-\frac{1}{2\beta}\, e^{-\beta x^2}\right]_0^\infty = \frac{1}{2\beta}$$

For $n>1$, I_n can be related to I_0 or I_1 with a reduction formula:

$$\begin{aligned} I_n &= \int_0^\infty x^{n-1}\, x\, e^{-\beta x^2}\,dx \\ &= \underbrace{\left[-x^{n-1}\,\frac{1}{2\beta}\, e^{-\beta x^2}\right]_0^\infty}_{0} + \frac{(n-1)}{2\beta}\underbrace{\int_0^\infty x^{n-2}\, e^{-\beta x^2}\,dx}_{I_{n-2}} \end{aligned}$$

Hence

$$I_n = \frac{(n-1)}{2\beta}\, I_{n-2} = \frac{(n-1)}{2\beta}\,\frac{(n-3)}{2\beta}\, I_{n-4} = \cdots$$

with the recursive chain continuing until we reach I_0 or I_1, depending on the parity of n. This leads to the result

$$I_n = \tfrac{(n-1)}{2\beta}\,\tfrac{(n-3)}{2\beta}\,\tfrac{(n-5)}{2\beta}\cdots\begin{cases}\cdots\tfrac{4}{2\beta}\,\tfrac{2}{2\beta}\,I_1 & \text{for } n \text{ odd}\\ \cdots\tfrac{3}{2\beta}\,\tfrac{1}{2\beta}\,I_0 & \text{for } n \text{ even}\end{cases}$$

Or, more compactly,

$$I_n = \frac{\left(\frac{n-1}{2}\right)!}{2\sqrt{\beta^{n+1}}} \quad \text{for } n \text{ odd} \qquad \text{and} \qquad I_n = \frac{n!}{(n/2)!\;2^{n+1}}\sqrt{\frac{\pi}{\beta^{n+1}}} \quad \text{for } n \text{ even}$$

These results enable the characteristic speeds of gas molecules to be calculated, because their distribution in equilibrium at temperature T is given by

$$\mathrm{f}(v) = A v^2 \mathrm{e}^{-\beta v^2} \quad \text{with } \beta = \frac{m}{2\,\mathrm{kT}}$$

where $v \geqslant 0$ is the speed, m is the mass of the gas particle, k is the Boltzmann constant, and A is the normalization factor ensuring that

$$\int_0^\infty \mathrm{f}(v)\,\mathrm{d}v = A\underbrace{\int_0^\infty v^2 \mathrm{e}^{-\beta v^2}\,\mathrm{d}v}_{I_2} = 1$$

$$A = 4\sqrt{\frac{\beta^3}{\pi}} = \frac{m}{\mathrm{kT}}\sqrt{\frac{2m}{\pi\,\mathrm{kT}}}$$

The average speed, $<v>$, is given by

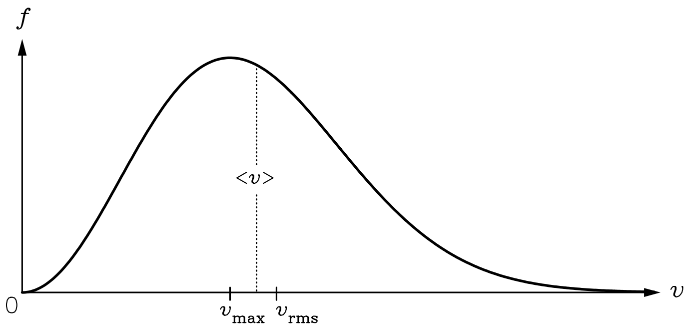
$$<v> = \int_0^\infty v\,\mathrm{f}(v)\,\mathrm{d}v = A\underbrace{\int_0^\infty v^3 \mathrm{e}^{-\beta v^2}\,\mathrm{d}v}_{I_3} = \frac{I_3}{I_2} = \frac{2}{\sqrt{\pi\beta}}$$

$$<v> = \sqrt{\frac{8\,\mathrm{kT}}{\pi m}}$$

while the root-mean-square speed, v_{rms}, is slightly higher:

$$v_{\mathrm{rms}}^2 = <v^2> = \int_0^\infty v^2\,\mathrm{f}(v)\,\mathrm{d}v = A\underbrace{\int_0^\infty v^4 \mathrm{e}^{-\beta v^2}\,\mathrm{d}v}_{I_4} = \frac{I_4}{I_2} = \frac{3}{2\beta}$$

$$v_{\mathrm{rms}} = \sqrt{\frac{3\,\mathrm{kT}}{m}}$$



$$\left.\frac{\mathrm{df}}{\mathrm{d}v}\right|_{v_{\mathrm{max}}} = 0$$

$$\Rightarrow \quad v_{\mathrm{max}} = \sqrt{\frac{2\,\mathrm{kT}}{m}}$$