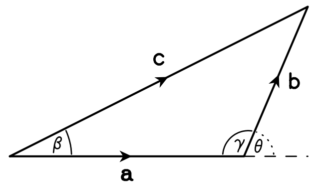
8 Vectors

(1) Use the cross product to obtain the 'sine rule' for triangles.

Consider the triangle formed by three vectors **a**, **b** and **c** such that $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

$$\therefore \quad \mathbf{a}\times\mathbf{c} \;=\; \mathbf{a}\times(\mathbf{a}+\mathbf{b}) \;=\; \underbrace{\mathbf{a}\times\mathbf{a}}_{0} + \mathbf{a}\times\mathbf{b} \;=\; \mathbf{a}\times\mathbf{b}$$



Defining β as the angle between **a** and **c**, and hence the angle opposite to the vector side **b**, and θ as the angle between **a** and **b**, we have

$$\left.\begin{aligned} \mathbf{a}\times\mathbf{b} \;&=\; |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta \\ \text{and}\quad \mathbf{a}\times\mathbf{b} \;&=\; \mathbf{a}\times\mathbf{c} \;=\; |\mathbf{a}|\,|\mathbf{c}|\,\sin\beta \end{aligned}\right\} \quad \therefore \quad |\mathbf{b}|\sin\theta \;=\; |\mathbf{c}|\sin\beta$$

If γ is the angle opposite vector side **c**, then the sum of γ and θ must be π radians, or 180°, and $\sin\gamma = \sin(\pi - \theta) = \sin\theta$.

$$\text{Hence}\quad \underline{\frac{|\mathbf{b}|}{\sin\beta} \;=\; \frac{|\mathbf{c}|}{\sin\gamma}}\;; \qquad \text{and, similarly,}\quad \mathbf{b}\times\mathbf{c} \;\Rightarrow\; \underline{\frac{|\mathbf{a}|}{\sin\alpha} \;=\; \frac{|\mathbf{c}|}{\sin\gamma}}$$

$$\alpha + \beta + \gamma = \pi$$

(2) Find the scalar triple product of the vectors $(1,2,4)$, $(2,0,-3)$ and $(-4,4,17)$. Are they linearly independent? Can the third vector be expressed as a linear combination of the first two? If so, find this linear combination.

$$(1,2,4)\bullet\big[(2,0,-3)\times(-4,4,17)\big] \;=\; (1,2,4)\bullet(12,-22,8) \;=\; \underline{0}$$

Since the scalar triple product of these three vectors is zero, we know that they must either be confined to a plane or lie along a single line. In such cases we say that the three vectors are linearly dependent, or <u>not</u> linearly independent. The vectors in question are not simple scalar multiples of each other, and so are not mutually parallel. They must therefore lie in a plane, so that the third vector <u>can</u> be expressed as a *linear combination* of the first two vectors:

$$(-4,4,17) \;=\; \alpha\,(1,2,4) + \beta\,(2,0,-3)$$

where α and β are constants. These coefficients can be evaluated by dotting both sides of the equation with $(3,0,2)$, a vector chosen to be perpendicular to $(2,0,-3)$:

$$(-4,4,17)\bullet(3,0,2) \;=\; \alpha\,(1,2,4)\bullet(3,0,2) + \beta\,(2,0,-3)\bullet(3,0,2)$$

$$\therefore\quad 22 \;=\; 11\,\alpha + 0 \quad\Rightarrow\quad \alpha = 2$$

Substituting $\alpha = 2$, and equating the x (or z) components, yields $\beta = -3$.

$$\text{i.e.}\quad \underline{(-4,4,17) \;=\; 2\,(1,2,4) - 3\,(2,0,-3)}$$