14 Partial differential equations

(1) The most general form of the solution to the one-dimensional wave equation is: $y(x,t) = \mathsf{G}(x+ct) + \mathsf{H}(x-ct)$, where G and H must be determined from suitable boundary conditions. Use this to solve the wave equation when

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0 \qquad \text{and} \qquad y(x,0) = \begin{cases} \mathsf{f}(x) & \text{for } 0<x<\mathsf{L} \\ 0 & \text{otherwise} \end{cases}$$

What happens if, additionally, (a) $y(0,t) = 0$ with $x \geqslant 0$; and (b) $y = 0$ at both $x = 0$ and $x = \mathsf{L}$ with $0 \leqslant x \leqslant \mathsf{L}$?

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Defining $u = x + ct$ and $v = x - ct$, so that $y = \mathsf{G}(u) + \mathsf{H}(v)$,

$$\delta y \approx \frac{\mathrm{d}\mathsf{G}}{\mathrm{d}u}\,\delta u + \frac{\mathrm{d}\mathsf{H}}{\mathrm{d}v}\,\delta v \quad \Rightarrow \quad \left(\frac{\partial y}{\partial t}\right)_x = \frac{\mathrm{d}\mathsf{G}}{\mathrm{d}u}\underbrace{\left(\frac{\partial u}{\partial t}\right)_x}_{c} + \frac{\mathrm{d}\mathsf{H}}{\mathrm{d}v}\underbrace{\left(\frac{\partial v}{\partial t}\right)_x}_{-c}$$

When $t = 0$, $u = v = x$ and the first boundary condition gives

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = c\left[\frac{\mathrm{d}\mathsf{G}}{\mathrm{d}x} - \frac{\mathrm{d}\mathsf{H}}{\mathrm{d}x}\right] = 0$$

Integration with respect to x then tells us that the functions G and H are the same to within an additive constant (K, say): $\mathsf{G}(x) = \mathsf{H}(x) + K$. Substituting this into the second $t = 0$ boundary condition,

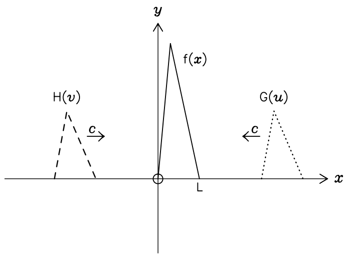
$$y(x,0) = \mathsf{G}(x) + \mathsf{H}(x) = 2\mathsf{H}(x) + K = \begin{cases} \mathsf{f}(x) & \text{for } 0<x<\mathsf{L} \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\mathsf{H}(x) = \tfrac{1}{2}\,\mathsf{f}(x) - \tfrac{K}{2} \qquad \text{and} \qquad \mathsf{G}(x) = \tfrac{1}{2}\,\mathsf{f}(x) + \tfrac{K}{2}$$

for $0 < x < \mathsf{L}$; otherwise, $\mathsf{H} = -K/2$ and $\mathsf{G} = K/2$. Replacing x with u in G, and v in H,

$$\mathsf{G}(x+ct) = \begin{cases} \tfrac{1}{2}\,\mathsf{f}(x+ct) & \text{for } -ct < x < \mathsf{L} - ct \\ 0 & \text{otherwise} \end{cases}$$



and

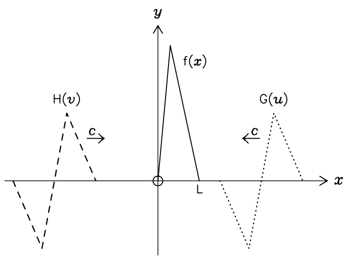
$$\mathsf{H}(x-ct) = \begin{cases} \tfrac{1}{2}\,\mathsf{f}(x-ct) & \text{for } ct < x < \mathsf{L}+ct \\ 0 & \text{otherwise} \end{cases}$$

We have omitted the arbitrary constant, K, because it cancels when G and H are added to obtain the solution $y(x,t) = \mathsf{G}(x+ct) + \mathsf{H}(x-ct)$.

(a) To satisfy the constraint that $y=0$ when $x=0$, for all time t, we require

$$y(0,t) = \mathsf{G}(ct) + \mathsf{H}(-ct) = 0$$

This means that G must be antisymmetric with respect to H. Earlier, however, we noted that the functional form of G and H was the same (to within an arbitrary additive constant that could be set to zero). Hence



$$\mathsf{H}(x) = \mathsf{G}(x) = -\mathsf{H}(-x)$$

To additionally ensure that $y(0,t) = 0$, therefore,

$$-\mathsf{H}(-x) = -\mathsf{G}(-x) = \mathsf{G}(x) = \mathsf{H}(x) = \begin{cases} \tfrac{1}{2}\,\mathsf{f}(x) & \text{for } 0 < x < \mathsf{L} \\ 0 & x \geqslant \mathsf{L} \end{cases}$$

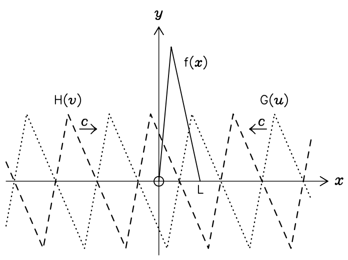
Then we simply replace x with u in G, and v in H, to obtain $y = \mathsf{G}(u) + \mathsf{H}(v)$.

(b) To also satisfy the constraint that $y=0$ when $x=\mathsf{L}$, we require

$$y(\mathsf{L},t) = \mathsf{G}(\mathsf{L}+ct) + \mathsf{H}(\mathsf{L}-ct) = 0$$

Since G and H have the same functional form, and are antisymmetric, they now need to be periodic as well:

$$y(\mathsf{L},t) = \mathsf{H}(\underbrace{\mathsf{L}+ct}_{\phi}) - \mathsf{H}(\underbrace{-\mathsf{L}+ct}_{\phi-2\mathsf{L}}) = 0$$



Hence

$$-\mathsf{G}(-u) = \mathsf{G}(u) = \tfrac{1}{2}\,\mathsf{f}(u) \quad \text{for} \quad 0 < u = x+ct < \mathsf{L}$$

and

$$-\mathsf{H}(-v) = \mathsf{H}(v) = \tfrac{1}{2}\,\mathsf{f}(v) \quad \text{for} \quad 0 < v = x-ct < \mathsf{L}$$

with $\mathsf{G}(u) = \mathsf{G}(u+2\mathsf{L})$ and $\mathsf{G}(v) = \mathsf{G}(v+2\mathsf{L})$.