Answer Key for Ch14 Exercise2: Show $\hat{\beta}$ is unbiased

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Show that the OLS estimate $\hat{\beta}_1$ is unbiased for the model $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$.

1. We first substitute the "true" equation for Y_i into our estimate of $\hat{\beta}_1$. Use the fact that $\overline{Y} = \beta_0 + \beta_1 \overline{X} + \overline{\epsilon}$ and that $\overline{\epsilon} = 0$ to generate the following. Note that the difference between β_1 and $\hat{\beta}_1$ is very important here.

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{\sum(\beta_0 + \beta_1 X_i + \epsilon_i - \beta_0 - \beta_1 \overline{X})(X_i - \overline{X})}{\sum(X_i - \overline{X})^2} \\
&= \frac{(\beta_1 \sum(X_i - \overline{X}) + \epsilon_i)(X_i - \overline{X})}{\sum(X_i - \overline{X})^2} \\
&= \frac{\beta_1 \sum(X_i - \overline{X})(X_i - \overline{X})}{\sum(X_i - \overline{X})^2} + \frac{\sum \epsilon_i (X_i - \overline{X})}{\sum(X_i - \overline{X})^2} \\
&= \beta_1 + \frac{\sum \epsilon_i (X_i - \overline{X})}{\sum(X_i - \overline{X})^2} \\
&= \beta_1 + \frac{\sum \epsilon_i X_i}{\sum(X_i - \overline{X})^2} - \frac{\overline{X} \sum \epsilon_i}{\sum(X_i - \overline{X})^2}
\end{aligned}
$$

The expectation of $\sum \epsilon_i = 0$ by assumption of OLS model. If ϵ_i is uncorrelated with X_i then the expectation of $\sum \epsilon_i X_i$ is also zero, meaning that

$$E[\hat{\beta}_1] = \beta_1$$

which means that the OLS estimate of β_1 is unbiased as long as the error is uncorrelated with the independent variable.