PART IV: Transforms and Fourier Series

Chapter 24: The Laplace transform

24.1. The Laplace transform of the function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t.$$

Use (24.2), (24.3) and (24.4).

(a)
$$\mathcal{L}\{\mathrm{e}^t\} = \int_0^\infty \mathrm{e}^t \mathrm{e}^{-st}\mathrm{d}t = \frac{1}{s-1}.$$

(b)
$$\mathcal{L}\{4\mathrm{e}^{-t}\} = \frac{4}{1+s}.$$

(c)
$$\mathcal{L}\{3\mathrm{e}^t - \mathrm{e}^{-t}\} = \mathcal{L}\{3\mathrm{e}^t\} - \mathcal{L}\{\mathrm{e}^{-t}\} = \frac{3}{s-1} - \frac{1}{s+1}.$$

(d)
$$\mathcal{L}\{3t^2 - 1\} = \frac{6}{s^3} - \frac{1}{s}.$$

(e)
$$\mathcal{L}\{\tfrac{1}{2}t^3 + 2t^2 - 3\} = \tfrac{3}{s^4} + \tfrac{4}{s^3} - \tfrac{3}{s}.$$

(f)
$$\mathcal{L}\{3 + 2t^4\} = \frac{3}{s} + \frac{48}{s^5}.$$

(g)
$$\mathcal{L}\{3\sin t - \cos t\} = \mathcal{L}\{3\sin t\} - \mathcal{L}\{\cos t\} = \frac{3}{1+s^2} - \frac{s}{1+s^2} = \frac{3-s}{1+s^2}.$$

(h)
$$\mathcal{L}\{2(\cos t - \sin t)\} = \mathcal{L}\{2\cos t\} - \mathcal{L}\{2\sin t\} = \frac{2s}{1+s^2} - \frac{2}{1+s^2}.$$

(i)
$$\mathcal{L}\left\{1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}\right\} = s + s^2 + s^3 + \cdots + s^{n+1}.$$

24.2. The scale rule (24.5) states that, if $\mathcal{L}\{f(t)\} = F(s)$ and $k > 0$, then

$$\mathcal{L}\{f(kt)\} = \frac{1}{k}F\left(\frac{s}{k}\right).$$

(a) Since $\mathcal{L}\{\mathrm{e}^t\} = 1/(s-1)$, then

$$\mathcal{L}\{\mathrm{e}^{3t}\} = \frac{1}{3}\frac{1}{(\frac{s}{3} - 1)} = \frac{1}{s-3}.$$

(b)
$$\mathcal{L}\{1-2\mathrm{e}^{-2t}\} = \frac{1}{s} - \frac{2}{s+2} = \frac{2-s}{s(s+2)}.$$

(c)
$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}.$$

(d)
$$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2}.$$

(e)
$$\mathcal{L}\{3\cos 2t - 2\sin 2t\} = \frac{3s}{4+s^2} - \frac{4}{4+s^2} = \frac{3s-4}{4+s^2}.$$

(f) Use the identity $\cos^2 t = \frac{1}{2}(1+\cos 2t)$. Then

$$\mathcal{L}\{\cos^2 t\} = \mathcal{L}\{\tfrac{1}{2}(1+\cos 2t)\} = \frac{2+s^2}{s(4+s^2)}.$$

(g) Use the identity $\sin^2 t = \frac{1}{2}(1-\cos 2t)$, so that

$$\mathcal{L}\{\sin^2 t\} = \frac{2}{s(4+s^2)}.$$

24.3. To evaluate these Laplace transforms use the scale, shift and other rules in Section 24.3.
(a) By (24.2), $\mathcal{L}\{t^2\} = 2!/s^3$. Then by the shift rule (24.7)

$$\mathcal{L}\{t^2\mathrm{e}^t\} = \frac{2!}{(s-1)^3}.$$

(b) In a way similar to (a), $\mathcal{L}\{t\} = 1/s^2$, so that, by the shift rule

$$\mathcal{L}\{t\mathrm{e}^{-2t}\} = \frac{1}{(s+2)^2}.$$

(c) Using the shift rule again

$$\mathcal{L}\{t^2\mathrm{e}^{-t}\} = \frac{2!}{(s+1)^2}.$$

(d) Since, by (24.4), $\mathcal{L}\{\cos t\} = s/(s^2+1)$, the shift rule (24.7) implies

$$\mathcal{L}\{e^{2t}\cos t\} = \frac{s-2}{(s-2)^2+1} = \frac{s-2}{s^2-4s+5}.$$

(e) Since, by (24.4), $\mathcal{L}\{\sin t\} = 1/(s^2+1)\}$, the shift rule (24.7) implies

$$\mathcal{L}\{\mathrm{e}^{-t}\sin t\} = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}.$$

(f) Using the scale rule (24.6) and the shift rule (24.7)

$$\mathcal{L}\{\mathrm{e}^{t}\sin 3t\} = \frac{3}{s^2-2s+10}.$$

(g) Using the scale rule (24.6) and the shift rule (24.7)

$$\mathcal{L}\{\mathrm{e}^{-2t}\sin 3t\} = \frac{3}{s^2+4s+13}.$$

(h) Using the scale rule (24.6) and the shift rule (24.7) again

$$\mathcal{L}\{e^{-3t}\cos 2t\} = \frac{3+s}{s^2+6s+13}.$$

(i) Let $F(s) = \mathcal{L}\{\cos 3t\} = s/(s^2+9)$. Using the multiplication rule

$$\mathcal{L}\{t\cos 3t\} = -\frac{d}{ds}\left(\frac{s}{s^2+9}\right) = \frac{s^2-9}{(s^2+9)^2}.$$

(j) Let $F(s) = \mathcal{L}\{\sin 3t\} = 3/(s^2+9)$. Now use the multiplication rule (24.8):

$$\mathcal{L}\{t\sin 3t\} = -\frac{dF(s)}{ds} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}.$$

(k) Let $F(s) = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$. By the multiplication rule (24.8):

$$\begin{aligned}\mathcal{L}\{t^2\sin t\} &= \frac{d^2F(s)}{ds^2} = \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)\\ &= -\frac{d}{ds}\left(\frac{2s}{(s^2+1)^2}\right)\\ &= \frac{6s^2-2}{(s^2+1)^3}.\end{aligned}$$

(l) Let $f(t) = t^4e^{-t}$.

(i) Since $\mathcal{L}\{t^4\} = 4!/s^5$, the shift rule (24.7) implies

$$\mathcal{L}\{f(t)\} = \frac{4!}{(s+1)^5} = \frac{24}{(s+1)^5}.$$

(ii) Since $\mathcal{L}\{e^{-t}\} = 1/(s+1) = F(s)$, say, the multiplication rule (24.8) gives

$$\begin{aligned}\mathcal{L}\{t^4e^{-t}\} &= (-1)^4\frac{d^4F(s)}{ds^4} = \frac{d^4}{ds^4}\left[\frac{1}{(s+1)}\right]\\ &= \frac{24}{(s+1)^5}\end{aligned}$$

(iii) The direct method requires repeated integration by parts:

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty t^4e^{-t}dt = [-t^4e^{-t}]_0^\infty + \int_0^\infty 4t^3e^{-t}dt\\ &= 0 + [-4t^3e^{-t}]_0^\infty + \int_0^\infty 12t^2e^{-t}dt\\ &= 12\int_0^\infty t^2e^{-t}dt = \frac{24}{(s+1)^5}.\end{aligned}$$

24.4. From the definition (24.1):

$$\int_0^\infty e^{-st}\cos kt\,dt = \frac{s}{s^2+k^2}.$$

Differentiate both sides of this equation with respect to k. Then

$$-\int_0^\infty te^{-st}\sin kt\,dt = -\frac{2ks}{(s^2+k^2)^2},$$

so that

$$\mathcal{L}\{t\sin kt\} = \frac{2ks}{(s^2+k^2)^2}.$$

24.5. In these problems we have to find the function $f(t)$ in the transform

$$F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t,$$

when $F(s)$ is given. We attempt this by using the tables in Section 24.4 and in Appendix F, and the various rules. Note that all the time functions are defined to be zero for $t < 0$.
(a) $F(s) = 1/s^2$ is the transform of $f(t) = t$ by (24.2). We also write this as

$$\frac{1}{s^2} \leftrightarrow t \text{ (see Section 24.4).}$$

(b)
$$\frac{1}{s} \leftrightarrow 1 \text{ (see 24.2).}$$

(c)
$$\frac{3}{2s} \leftrightarrow \frac{3}{2}.$$

(d)
$$\frac{3}{s^5} \leftrightarrow \frac{t^4}{8}.$$

(e) Use the shift rule (24.7) $\mathcal{L}\{\mathrm{e}^{kt}f(t)\} = F(s-k)$. Since $\mathcal{L}\{1\} = 1/s$,

$$\frac{1}{s-3} \leftrightarrow \mathrm{e}^{3t}.$$

(f) Using the shift rule again

$$\frac{1}{s+4} \leftrightarrow \mathrm{e}^{-4t}.$$

(g)
$$\frac{3}{2s-1} \leftrightarrow \frac{3}{2}\mathrm{e}^{\frac{1}{2}t}.$$

(h)
$$\frac{2}{2-3s} \leftrightarrow -\frac{2}{3}\mathrm{e}^{\frac{2}{3}t}.$$

(i) Using partial fractions

$$\frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s} \leftrightarrow \mathrm{e}^t - 1.$$

(j) Since $s^2 + s - 1 = (s + \frac{1}{2} - \frac{1}{2}\sqrt{5})(s + \frac{1}{2} + \frac{1}{2}\sqrt{5})$, the partial fractions expansion is

$$\begin{aligned} \frac{1}{s^2+s-1} &= \frac{1}{\sqrt{5}}\left[\frac{1}{s+\frac{1}{2}-\frac{1}{2}\sqrt{5}} - \frac{1}{s+\frac{1}{2}+\frac{1}{2}\sqrt{5}}\right] \\ &\leftrightarrow \frac{1}{\sqrt{5}}\left[\mathrm{e}^{-\frac{1}{2}(1-\sqrt{5})t} - \mathrm{e}^{-\frac{1}{2}(1+\sqrt{5}t)}\right] \end{aligned}$$

(k) Using partial fractions

$$\frac{s}{s^2-1} = \frac{1}{2(s+1)} + \frac{1}{2(s-1)} \leftrightarrow \frac{1}{2}[\mathrm{e}^{-t} + \mathrm{e}^t].$$

(l) Using partial fractions

$$\frac{2s-1}{s^2-1} = \frac{1}{2(s-1)} + \frac{3}{2(s+1)} \leftrightarrow \frac{1}{2}\mathrm{e}^t + \frac{3}{2}\mathrm{e}^{-t}.$$

(m) By (24.4)

$$\frac{s}{s^2+1} \leftrightarrow \cos t.$$

(n) By (24.6c),

$$\frac{1}{s^2+4} \leftrightarrow \frac{1}{2}\sin 2t.$$

(o) By (24.6),

$$\frac{2s-1}{s^2+4} = \frac{2s}{s^2+4} - \frac{1}{s^2+4} \leftrightarrow 2\cos 2t - \frac{1}{2}\sin 2t.$$

(p) Using partial fractions

$$\frac{2s-1}{s(s-1)} = \frac{1}{s} + \frac{1}{s-1} \leftrightarrow 1 + \mathrm{e}^t.$$

(q) Using partial fractions

$$\begin{aligned} \frac{s^2-1}{s(s-1)(s+2)(s+3)} &= \frac{1}{6s} + \frac{1}{2(s+2)} - \frac{2}{3(s+3)} \\ &\leftrightarrow \frac{1}{6} + \frac{1}{2}\mathrm{e}^{-2t} - \frac{2}{3}\mathrm{e}^{-3t} \end{aligned}$$

(r) Using partial fractions and (24.6)

$$\begin{aligned} \frac{s}{(s-1)(s^2+1)} &= \frac{1}{2(s-1)} - \frac{s}{2(1+s^2)} + \frac{1}{2(1+s^2)} \\ &\leftrightarrow \frac{1}{2}\mathrm{e}^t - \frac{1}{2}\cos t + \frac{1}{2}\sin t \end{aligned}$$

(s) Since $\mathcal{L}\{t^2\} = 2/s^3$, the shift rule (24.7) applied to this transform implies

$$\frac{1}{(s-1)^3} \leftrightarrow \frac{1}{2}\mathrm{e}^t t^2.$$

(t) Since $s^2 - 2s + 2 = (s-1)^2 + 1$, we can write the transform as

$$\frac{2s+1}{s^2-2s+1} = \frac{2(s-1)+3}{(s-1)^2+1}.$$

Now use the shift rule (24.7) and (24.6b,c):

$$\frac{2s+1}{s^2-2s+1} \leftrightarrow \frac{1}{2}\mathrm{e}^t[\cos t + 6\sin t].$$

(u) Using partial fractions

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{s}{3(s^2+1)} - \frac{s}{3(s^2+4)} \leftrightarrow \frac{1}{3}[\cos t - \cos 2t].$$

24.6. These problems use the results on the transforms of derivatives given in (24.12):

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0), \qquad \mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx(0) - \dot{x}(0).$$

(a)

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0) = sX(s) - 6.$$

(b) $$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0) = sX(s).$$

(c) $$\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx(0) - \dot{x}(0) = s^2X(s) - 3s - 5.$$

(d) $$\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx(0) - \dot{x}(0) = s^2X(s).$$

(e)

$$\begin{aligned}\mathcal{L}\{2\ddot{x}(t) + 3\dot{x}(t) - 2x(t)\} &= 2s^2X(s) - 2sx(0) - 2\dot{x}(0) + 3sX(s) - 3x(0) - 2X(s) \\ &= (2s^2 + 3s - 2)X(s) - (3 + 2s)x(0) - 2\dot{x}(0) \\ &= (2s^2 + 3s - 2)X(s) - 10s - 9\end{aligned}$$

(f) $$\mathcal{L}\{3\ddot{x}(t) - 5\dot{x}(t) + x(t) - 1\} = (3s^2 - 5s + 1)X(s) - \frac{1}{s}.$$

24.7. (a) Take the Laplace transform of the differential equation:

$$\begin{aligned}\mathcal{L}\{\ddot{x} + 3\dot{x} + 2x\} &= s^2X(s) - sx(0) - \dot{x}(0) + 3sX(s) - 3x(0) + 2X(s) \\ &= (s^2 + 3s + 2)X(s) - 1 = 0.\end{aligned}$$

Therefore

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Taking the inverse transform $x(t) = \mathrm{e}^{-t} - \mathrm{e}^{-2t}$.

(b) Take the Laplace transform of the equation:

$$\begin{aligned}\mathcal{L}\{\ddot{x} + \dot{x} - 2x\} &= s^2X(s) - sx(0) - \dot{x}(0) + sX(s) - x(0) - 2X(s) \\ &= (s^2 + s - 2)X(s) - 3(s + 1) = 0.\end{aligned}$$

Hence the transform of the solution is

$$X(s) = \frac{3(s+1)}{s^2 + s - 2} = \frac{3s}{(s-1)(s+2)} = \frac{2}{s-1} + \frac{1}{s+2}.$$

Hence the inverse transform is $x(t) = 2\mathrm{e}^{t} + \mathrm{e}^{-2t}$.

(c) Take the Laplace transform of the equation:

$$\begin{aligned}\mathcal{L}\{\ddot{x} + 4\dot{x}\} &= s^2X(s) - sx_0 - y_0 + 4sX(s) - 4x_0 \\ &= (s^2 + 4s)X(s) - (s + 4)x_0 - y_0 = 0\end{aligned}$$

Therefore the transform of the solution is

$$X(s) = \frac{(s+4)x_0 + y_0}{s(s+4)} = \frac{4x_0 + y_0}{4s} - \frac{y_0}{4(s+4)}.$$

Hence the solution is

$$x = \tfrac{1}{4}(4x_0 + y_0) - \tfrac{1}{4}y_0\mathrm{e}^{-4t}.$$

(d) Take the Laplace transform of the equation:

$$\mathcal{L}\{\ddot{x} + \omega^2 x\} = s^2X(s) - sx(0) - \dot{x}(0) + \omega^2X(s) = (s^2 + \omega^2)X(s) - sc = 0.$$

Therefore the transform of the solution is

$$X(s) = \frac{sc}{s^2 + \omega^2}$$

giving the solution $x = c\cos\omega t$.

(e) Taking the Laplace transform of the equation:

$$\begin{aligned}\mathcal{L}\{\ddot{x}+2\dot{x}+5x\} &= s^2X(s)-3s+3+2sX(s)-6+5X(s)\\ &= (s^2+2s+5)X(s)-3s-3=0.\end{aligned}$$

Therefore

$$X(s)=\frac{3s+3}{s^2+2s+5}=\frac{3(s+1)}{(s+1)^2+4}.$$

Using the shift rule

$$x=3\mathrm{e}^{-t}\cos 2t.$$

(f) For this fourth-order equation

$$s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)-Y(s)=(s^4-1)Y(s)-s^3=0.$$

Therefore

$$Y(s)=\frac{s^3}{s^4-1}=\frac{1}{4(s-1)}+\frac{1}{4(s+1)}+\frac{s}{2(s^2+1)}.$$

The solution can now be constructed using (24.6):

$$y=\tfrac{1}{4}\mathrm{e}^{x}+\tfrac{1}{4}\mathrm{e}^{-x}+\tfrac{1}{2}\cos x.$$

24.8. Take the Laplace transform of the equation in each case.

(a) $\ddot{x}=1+t+\mathrm{e}^t$, $x(0)=\dot{x}(0)=0$. Then

$$s^2X(s)=\frac{1}{s}+\frac{1}{s^2}+\frac{1}{s-1}.$$

Hence

$$X(s)=\frac{1}{s^3}+\frac{1}{s^4}+\frac{1}{s^2(s-1)}=\frac{1}{s^3}+\frac{1}{s^4}-\frac{1}{s}-\frac{1}{s^2}+\frac{1}{s-1}.$$

Hence, using the table of Laplace transforms, the solution is

$$x=-1-t+\frac{1}{2}t^2+\frac{1}{6}t^3+\mathrm{e}^t.$$

(b) $\ddot{x}+x=3$, $x(0)=0$, $\dot{x}(0)=1$. The Laplace transform of this equation is

$$s^2X(s)-1+X(s)=\frac{3}{s}.$$

Hence

$$X(s)=\frac{1}{s^2+1}+\frac{3}{s(s^2+1)}=\frac{1}{s^2+1}+\frac{3}{s}-\frac{3s}{s^2+1}.$$

Using (24.6) for the inversion:

$$x=\sin t+3-3\cos t.$$

(c) $\ddot{x}+2\dot{x}+2x=3$, $x(0)=1$, $\dot{x}(0)=0$. The Laplace transform of the system is

$$s^2X(s)-s+2sX(s)-2+2X(s)=\frac{3}{s}.$$

Hence

$$\begin{aligned}X(s) &= \frac{s+2}{(s+1)^2+1}+\frac{3}{s[(s+1)^2+1]}\\ &= \frac{s+2}{(s+1)^2+1}+\frac{3}{2s}-\frac{3(s+2)}{2[(s+1)^2+1]}\\ &= -\frac{s+1}{2[(s+1)^2+1]}-\frac{1}{2[(s+1)^2+1]}+\frac{3}{2s}.\end{aligned}$$

Hence using the inverse table and the shift rule

$$x = -\frac{1}{2}\mathrm{e}^{-t}(\cos t + \sin t) + \frac{3}{2}.$$

(d) $\ddot{x} - x = \mathrm{e}^{2t}$, $x(0) = 0$, $\dot{x}(0) = 1$. The Laplace transform of the system is

$$s^s X(s) - 1 - X(s) = \frac{1}{s-2}.$$

Hence

$$X(s) = \frac{s-1}{(s-2)(s^2-1)} = \frac{1}{(s-2)(s+1)} = \frac{1}{3(s-2)} - \frac{1}{3(s+1)},$$

using partial fractions. By inverting, we have

$$x = \frac{1}{3}(-\mathrm{e}^{-t} + \mathrm{e}^{2t}).$$

(e) $\ddot{x} - x = t\mathrm{e}^{t}$, $x(0) = 1$, $\dot{x}(0) = 1$. The Laplace transform of the system is

$$s^2 X(s) - s - 1 - X(s) = \frac{1}{(s-1)^2}.$$

Hence

$$X(s) = \frac{1}{s-1} + \frac{1}{(s-1)^2(s^2-1)} = \frac{9}{8(s-1)} - \frac{1}{8(s+1)} - \frac{1}{4(s-1)^2} + \frac{1}{2(s-1)^3}.$$

The solution is

$$x = \tfrac{9}{8}\mathrm{e}^{t} - \tfrac{1}{8}\mathrm{e}^{-t} - \tfrac{1}{4}t\mathrm{e}^{t} + \tfrac{1}{4}t^2\mathrm{e}^{t}.$$

(f) $\ddot{x} - 4x = 1 - \mathrm{e}^{2t}$, $x(0) = 1$, $\dot{x}(0) = -1$. The Laplace transform of the system is

$$s^2 X(s) - s + 1 - 4X(s) = \frac{1}{s} - \frac{1}{s-2}.$$

Hence

$$X(s) = -\frac{1}{4(s-2)^2} + \frac{7}{16(s-2)} - \frac{1}{4s} + \frac{13}{16(s+2)}.$$

Inverting

$$x = -\frac{1}{4}t\mathrm{e}^{2t} + \frac{7}{16}\mathrm{e}^{2t} - \frac{1}{4} + \frac{13}{16}\mathrm{e}^{-2t}.$$

(g) $\ddot{x} - 4x = \mathrm{e}^{2t} + \mathrm{e}^{-2t}$, $x(0) = 0$, $\dot{x}(0) = 0$. The Laplace transform of the system is

$$s^2 X(s) - 4X(s) = \frac{1}{s-2} + \frac{1}{s+2}.$$

Hence

$$X(s) = \frac{1}{(s^2-4)(s-2)} + \frac{1}{(s^2-4)(s+2)} = \frac{1}{4(s-2)^2} - \frac{1}{4(s+2)^2}.$$

Using the shift rule

$$x = \frac{t}{4}(\mathrm{e}^{2t} - \mathrm{e}^{-2t}).$$

(h) $\ddot{x} + \omega^2 x = C\cos\omega t$, $x(0) = x_0$, $\dot{x}(0) = y_0$. The Laplace transform of the equation is

$$s^2 X(s) - sx_0 - y_0 + \omega^2 X(s) = \frac{Cs}{s^2+\omega^2}.$$

Hence

$$X(s) = \frac{x_0 s + y_0}{(s^2+\omega^2)} + \frac{Cs}{(s^2+\omega^2)^2}.$$

Note that

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2} = F(s),$$

say, so that by (24.8)

$$-\frac{\mathrm{d}X(s)}{\mathrm{d}s} = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{2s\omega}{(s^2+\omega^2)^2} = \mathcal{L}\{t\sin\omega t\}.$$

Use this result in the inversion the transform above:

$$x = x_0\cos\omega t + \frac{y_0}{\omega}\sin\omega t + \frac{Ct}{2\omega}\sin\omega t.$$

(i) $\dddot{x} - 2\ddot{x} - \dot{x} + 2x = \mathrm{e}^{-2t}$, $x(0) = \dot{x}(0) = 0$, $\ddot{x}(0) = 2$. The Laplace transform of the equation is

$$s^3X(s) - 2 - 2s^2X(s) - sX(s) + 2X(s) = \frac{1}{s+2},$$

or

$$(s^3 - 2s^2 - s + 2)X(s) = (s-1)(s+1)(s-2)X(s) = 2 + \frac{1}{s+2}.$$

Hence, applying partial fractions,

$$\begin{aligned} X(s) &= \frac{2}{(s-1)(s+1)(s-2)} + \frac{1}{(s-1)(s+1)(s-2)(s+2)} \\ &= \frac{3}{4(s-2)} - \frac{7}{6(s-1)} - \frac{1}{2(s+1)} - \frac{1}{12(s+2)}. \end{aligned}$$

Inverting

$$x = \frac{3}{4}\mathrm{e}^{2t} - \frac{7}{6}\mathrm{e}^{t} - \frac{1}{2}\mathrm{e}^{-t} - \frac{1}{12}\mathrm{e}^{-2t}.$$

24.9. (a) $\dot{x} = x - y$, $\dot{y} = x + y$, $x(0) = 1$, $y(0) = 0$. Let $\mathcal{L}\{x\} = X(s)$ and $\mathcal{L}\{Y\} = Y(s)$. Taking Laplace transforms of the differential equations:

$$sX(s) - x(0) = sX(s) - 1 = X(s) + Y(s), \quad sY(s) - y(0) = sY(s) = X(s) + Y(s),$$

which are simultaneous linear equations in $X(s)$ and $Y(s)$, namely

$$(s-1)X(s) + Y(s) = 1, \quad X(s) + (s-1)Y(s) = 0.$$

Solving them

$$X(s) = \frac{s-1}{(s-1)^2+1}, \quad Y(s) = \frac{s-1}{(s-1)^2+1}.$$

Finally invert the transforms using the shift rule (24.7) applied to (24.4) to give the solutions

$$x = \mathrm{e}^t\cos t, \quad y = \mathrm{e}^t\sin t.$$

(b) $\dot{x} = 2x + 4y + \mathrm{e}^{4t}$, $\dot{y} = x + 2y$, $x(0) = 1$, $y(0) = 0$. Taking Laplace transforms of the differential equations:

$$sX(s) - 1 = 2X(s) + 4Y(s) + \frac{1}{s-4}, \quad sY(s) = X(s) + 2Y(s),$$

or

$$(s-2)X(s) - 4Y(s) = 1 + \frac{1}{s-4}, \quad X(s) + (s-2)Y(s) = 0.$$

The solutions are

$$X(s) = \frac{s-2}{s(s-4)} + \frac{s-2}{s(s-4)^2}, \quad Y(s) = \frac{1}{s(s-4)} + \frac{1}{s(s-4)^2}.$$

Using partial fractions

$$X(s) = \frac{3}{8s} + \frac{5}{8(s-4)} + \frac{1}{2(s-4)^2}, \quad Y(s) = -\frac{3}{16s} + \frac{3}{16(s-4)} + \frac{1}{4(s-4)^2}.$$

Finally, using (24.6a) and the rules (24.7) and (24.8):

$$x = \frac{3}{8} + \frac{5}{8}\mathrm{e}^{4t} + \frac{1}{2}t\mathrm{e}^{4t}, \quad y = -\frac{3}{16} + \frac{3}{16}\mathrm{e}^{4t} + \frac{1}{4}t\mathrm{e}^{4t}.$$

(c) $\dot{x} = x - 4y$, $\dot{y} = x + 2y$, $x(0) = 2$, $y(0) = 1$. Taking Laplace transforms of the differential equations:

$$sX(s) - 2 = X(s) - 4Y(s), \quad sY(s) - 1 = X(s) + 2Y(s),$$

or

$$(s-1)X(s) + 4Y(s) = 2, \quad X(s) - (s-2)Y(s) = -1.$$

Solving for $X(s)$ and

$$X(s) = \frac{2s-8}{s^2 - 3s + 6} = \frac{2s-8}{(s-\frac{3}{2})^2 + \frac{15}{4}}, \quad Y(s) = \frac{s+1}{s^4 - 3s^2 + 6} = \frac{s+1}{(s-\frac{3}{2})^2 + \frac{15}{4}}.$$

These transforms can be inverted using the shift rule and (24.6b,c):

$$x = -\frac{2}{3}\mathrm{e}^{\frac{3t}{2}}\left[-3\cos\left(\frac{\sqrt{15}t}{2}\right) + \sqrt{15}\sin\left(\frac{\sqrt{15}t}{2}\right)\right],$$

$$y = \frac{1}{3}\mathrm{e}^{\frac{3t}{2}}\left[3\cos\left(\frac{\sqrt{15}t}{2}\right) + \sqrt{15}\sin\left(\frac{\sqrt{15}t}{2}\right)\right].$$

24.10. (a) $\ddot{x} + x = \mathrm{e}^t$, $x(0) = A$, $\dot{x}(0) = B$. Take the Laplace transform of the equation:

$$s^2X(s) - As - B = \frac{1}{s-1}.$$

Hence

$$X(s) = \frac{As+B}{s^2+1} + \frac{1}{(s-1)(s^2+1)} = \frac{(A-\frac{1}{2})s + (B-\frac{1}{2})}{s^2+1} + \frac{1}{2(s-1)}.$$

Inverting using (24.6)

$$x = (A - -\tfrac{1}{2})\cos t + (B - \tfrac{1}{2})\sin t + \tfrac{1}{2}\mathrm{e}^t.$$

(b) $\ddot{x} - x = 3$, $x(0) = A$, $\dot{x}(0) = B$. The Laplace transform of the equation is:

$$s^2X(s) - As - B - X(s) = \frac{3}{s}.$$

Hence

$$X(s) = \frac{As+B}{s^2-1} + \frac{3}{s(s^2-1)} = \frac{3+A+B}{2(s-1)} - \frac{3}{s} + \frac{3+A-B}{2(s+1)}.$$

using partial fractions. Inversion gives the solution

$$x = -3 + \tfrac{1}{2}(3+A-B)\mathrm{e}^{-t} + \tfrac{1}{2}(3+A+B)\mathrm{e}^t.$$

(c) $\ddot{x} - 2\dot{x} + x = \mathrm{e}^t$, $x(0) = A$, $\dot{x}(0) = B$. The Laplace transform of the equation is:

$$s^2X(s) - As - B - 2sX(s) - A + X(s) = \frac{1}{s-1}.$$

Hence

$$X(s) = \frac{As+B-2A}{(s-1)^2} + \frac{1}{(s-1)^3} = \frac{A}{s-1} + \frac{B-A}{(s-1)^2} + \frac{1}{(s-1)^3}.$$

Inversion gives the solution

$$x = Ae^t + (B - A)te^t + \tfrac{1}{2}t^2e^t.$$

24.11. $\mathrm{d}^4y/\mathrm{d}x^4 - y = \mathrm{e}^x$, $y(0) = A$, $y'(0) = B$, $y''(0) = C$, $y'''(0) = D$. The Laplace transform of the equation is (with $\mathcal{L}y(x) = Y(s)$)

$$s^4Y(s) - As^3 - Bs^2 - Cs - D - Y(s) = \frac{1}{s-1}.$$

Therefore, after a lengthy partial fraction expansion,

$$\begin{aligned} Y(s) &= \frac{As^3 + Bs^2 + Cs + D}{(s^4-1)} + \frac{1}{(s-1)(s^4-1)} \\ &= \frac{1}{4(s-1)^2} + \frac{-3 + 2A + 2B + 2C + 2D}{8(s-1)} + \frac{1 + 2A - 2B + 2C - 2D}{8(s+1)} \\ &\quad + \frac{1 + 2B - 2D + (1 + 2A - 2C)s}{4(s^2+1)}. \end{aligned}$$

The inversion gives

$$\begin{aligned} y &= \frac{1}{4}x\mathrm{e}^x + \frac{1}{8}(2A + 2B + 2C + 2D - 3)\mathrm{e}^x + \frac{1}{8}(1 + 2A - 2B + 2C - 2D)\mathrm{e}^{-x} \\ &\quad + \frac{1}{4}(1 + 2A - 2C)\cos x + \frac{1}{4}(1 + 2B - 2D)\sin x. \end{aligned}$$

24.12. The equation for $x_0(t)$ has a different form from the subsequent equations; its transform using $x_0 = 1$ is

$$sX_0(s) - 1 + \beta X_0(s) = 0,$$

so that

$$X_0(s) = \frac{1}{s+\beta}, \text{ and } x_0(t) = \mathrm{e}^{-\beta t}. \tag{i}$$

For all $r \geq 1$, the form of the equation is

$$x_r + \beta x_r = \beta x_{r-1},$$

with $x_r(0) = 0$. Therefore the transform is

$$sX_r(s) + \beta X_r(s) = \beta X_{r-1}(s),$$

so that

$$X_r(s) = \frac{\beta}{s+\beta}X_{r-1}(s). \tag{ii}$$

Starting with the case $r = 1$, and using (i), we obtain the sequence

$$X_1(s) = \frac{\beta}{(s+\beta)^2}, \quad X_2(s) = \frac{\beta}{s+\beta}\frac{\beta}{(s+\beta)^2} = \frac{\beta^2}{(s+\beta)^3},$$

$$X_3(s) = \frac{\beta}{s+\beta}\frac{\beta^2}{(s+\beta)^3} = \frac{\beta^3}{(s+\beta)^4}, \ldots,$$

and, in general, for $r \geq 1$,

$$X_r(s) = \frac{\beta^r}{(s+\beta)^{r+1}}. \tag{iii}$$

From (24.2),

$$\frac{1}{s^{r+1}} \leftrightarrow \frac{1}{r!}.$$

From the shift rule (24.7), with $k = -\beta$, we obtain the inverse of (iii),

$$x_r(t) = \frac{\beta^r t^r}{r!} \mathrm{e}^{-\beta t}, \quad r \geq 1.$$

Together with (i), this provides the required solution.

24.13. By the multiplication rule (24.8),

$$t \cos t \mathrm{H}(t) \leftrightarrow -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}.$$

By the delay rule (24.15), with $c = 2$,

$$(t-2)\cos(t-2)\mathrm{H}(t-2) \leftrightarrow \frac{\mathrm{e}^{-2s}(s^2-1)}{(s^2+1)^2}.$$

By the shift rule (24.7), with $k = -1$,

$$\mathrm{e}^{-t}(t-2)\cos(t-2)\mathrm{H}(t-2) \leftrightarrow \frac{\mathrm{e}^{-2(s+1)}[(s+1)^2-1]}{[(s+1)^2+1]} = \frac{\mathrm{e}^{-2(s+1)}s(s+2)}{(s^2+2s+2)^2}.$$

24.14. (a) $G(s) = \mathrm{e}^{-2s}/(s+3)$. By (24.6a),

$$\frac{1}{s+3} \leftrightarrow \mathrm{e}^{-3t}.$$

Then by (24.15)

$$\frac{\mathrm{e}^{-2s}}{s+3} = \mathrm{e}^{-3(t-2)}\mathrm{H}(t-2).$$

(b) $G(s) = (1 - s\mathrm{e}^{-s})/(s^2+1)$. $G(s)$ is the sum of two transforms. For the first

$$\frac{1}{s^2+1} \leftrightarrow \sin t.$$

For the second start with

$$\frac{s}{s^2+1} \leftrightarrow \cos t.$$

Using (24.15),

$$\frac{s\mathrm{e}^{-2s}}{s^2+1} \leftrightarrow \cos(t-1)\mathrm{H}(t-1).$$

Finally, taking the difference

$$G(s) = (1 - s\mathrm{e}^{-s})/(s^2+1) \leftrightarrow \sin t - \cos(t-1)\mathrm{H}(t-1).$$

(c)

$$\mathrm{e}^{-2s}/(s-4) \leftrightarrow e^{4t-8}\mathrm{H}(t-2).$$

(d) $G(s) = s\mathrm{e}^{-s}/[(s+1)(s+2)]$. The e^{-s} term indicates that the delay rule will apply. Using partial fractions

$$\frac{s}{(s+1)(s+2)} = -\frac{1}{s+1} + \frac{2}{s+2} \leftrightarrow -\mathrm{e}^{-t} + 2\mathrm{e}^{-2t}.$$

The delay rule (24.15) gives the required function:

$$(-\mathrm{e}^{1-t} + 2\mathrm{e}^{2-2t})\mathrm{H}(t-1).$$

(e)

$$\frac{\mathrm{e}^{-s}}{(s-1)(s^2-2s+2)} \leftrightarrow \mathrm{e}^{t-1}[\cos(t+1)].$$

24.15. All problems have the same initial conditions $x(0) = \dot{x}(0) = 0$.
(a) $\ddot{x} + x = f(t)$, where

$$f(t) = \mathrm{H}(1-t) = \begin{cases} 1 & \text{for } 0 < t \leq 1, \\ 0 & \text{for } t > 1. \end{cases}.$$

Take the Laplace transform of the equation with the initial conditions $x(0) = \dot{x}(0) = 0$:

$$s^2 X(s) + X(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t = \int_0^1 \mathrm{e}^{-st}\mathrm{d}t = \frac{1-\mathrm{e}^{-s}}{s}.$$

Hence

$$X(s) = \frac{1-\mathrm{e}^{-s}}{s(s^2+1)} = (1-\mathrm{e}^{-s})\left[\frac{1}{s} - \frac{s}{s^2+1}\right].$$

Inverting using the delay rule:

$$x = 1 - \cos t - [1 - \cos(1-t)]\mathrm{H}(t-1).$$

(b) $\ddot{x} - 4x = f(t)$, where

$$f(t) = \begin{cases} 1 & \text{for } 0 < t \leq 1, \\ 0 & \text{for } t > 1. \end{cases} = \mathrm{H}(1-t).$$

Taking Laplace transforms:

$$s^2 X(s) - 4X(s) = \frac{1-\mathrm{e}^{-s}}{s},$$

as in (a). Hence, using partial fractions,

$$X(s) = \frac{1-\mathrm{e}^{-s}}{s(s^2-4)} = \left[\frac{1}{8(s-2)} + \frac{1}{8(s+2)} - \frac{1}{4s}\right] - \mathrm{e}^{-s}\left[\frac{1}{8(s-2)} + \frac{1}{8(s+2)} - \frac{1}{4s}\right].$$

Inverting

$$x = \frac{1}{8}\mathrm{e}^{2t} + \frac{1}{8}\mathrm{e}^{-2t} - \frac{1}{4} - \left[\frac{1}{8}\mathrm{e}^{2t-2} + \frac{1}{8}\mathrm{e}^{-(2t-2)} + \frac{1}{4}\right]\mathrm{H}(t-1).$$

(c) $\ddot{x} - 4x = f(t)$, where

$$f(t) = \begin{cases} t & \text{for } 0 < t \leq 1, \\ 2-t & \text{for } 1 < t \leq 2, \\ 0 & \text{for } t > 2. \end{cases}$$

In terms of step functions

$$f(t) = (2-t)\mathrm{H}(2-t) - (2-2t)\mathrm{H}(1-t).$$

Taking Laplace transforms

$$\begin{aligned} s^2 X(s) - 4X(s) &= \mathcal{L}\{f(t)\} = \int_0^2 (2-t)\mathrm{e}^{st}\mathrm{d}t - \int_0^1 (2-2t)\mathrm{e}^{-st}\mathrm{d}t \\ &= \left[\frac{\mathrm{e}^{-2s}}{s^2} - \frac{1-2s}{s^2}\right] - \left[\frac{2\mathrm{e}^{-s}}{s^2} - \frac{2(1-s)}{s^2}\right] \\ &= \frac{1-2\mathrm{e}^{-s}+\mathrm{e}^{-2s}}{s^2} \end{aligned}$$

Hence

$$X(s) = \frac{1-2\mathrm{e}^{-s}+\mathrm{e}^{-2s}}{s^2(s^2-4)} = (1-2\mathrm{e}^{-s}+\mathrm{e}^{-2s})\left[-\frac{1}{4s^2} + \frac{1}{16(s-2)} - \frac{1}{16(s+2)}\right].$$

Inverting this transform using the delay rule (24.15),

$$x = -\frac{1}{16}e^{-2t} + \frac{1}{16}e^{2t} - \frac{t}{4} + \left[-\frac{1}{16}e^{4-2t} + \frac{1}{2} + \frac{1}{16}e^{2t-4} - \frac{t}{4}\right]H(t-2)$$
$$+ \left[\frac{1}{8}e^{2-2t} - \frac{1}{2} - \frac{1}{8}e^{2t-2} + \frac{t}{2}\right]H(t-1)$$

(d) $\ddot{x} + x = f(t)$, where

$$f(t) = \begin{cases} \cos t & \text{for } 0 < t \le \pi, \\ 0 & \text{for } t > \pi. \end{cases}$$

so that $f(t) = \cos t[H(t) - H(t-\pi)]$. Take the Laplace transform of the equation:

$$s^2X(s) + X(s) = \frac{(1 + e^{-\pi s})s}{s^2 + 1}.$$

Hence

$$X(s) = \frac{(1 + e^{-\pi s})s}{s^2 + 1)^2}.$$

The inversion gives the solution

$$x = -\frac{1}{2}t\sin t - \frac{1}{2}(t-\pi)H(t-\pi)\sin t.$$

Chapter 25: Laplace and z transforms: applications

25.1. Use the division rule (25.1) which states that, if $G(s) \leftrightarrow g(t)$, then $G(s)/s \leftrightarrow \int_0^t g(\tau)d\tau$. Use also Table (24.10) of inverse transforms

(a) Since

$$\frac{1}{s^2+1} \leftrightarrow \sin t,$$

then

$$\frac{1}{s(s^2+1)} \leftrightarrow \int_0^t \sin\tau d\tau = [-\cos\tau]_0^t = 1 - \cos t.$$

(b) The division rule is applied to the result from (a) as follows:

$$\frac{1}{s^2(s^2+1)} \leftrightarrow \int_0^t \tau(1-\cos\tau)d\tau = [\tau - \sin\tau]_0^t = t - \sin t.$$

(c) Apply the division rule again to the result in (b):

$$\frac{1}{s^3(s^2+1)} \leftrightarrow \int_0^\tau (\tau - \sin\tau)d\tau = [\tfrac{1}{2}\tau^2 + \cos\tau]_0^t = \tfrac{1}{2}t^2 + \cos t - 1$$

25.2. The RLC circuit has the equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t).$$

Take the Laplace transform of the equation and use the division rule (25.1):

$$L[sI(s) - I(0)] + RI(s) + \frac{1}{Cs}I(s) = V(s),$$

where $i(t) \leftrightarrow I(s)$ and $v(t) \leftrightarrow V(s)$. Solving for $I(s)$, we obtain

$$I(s) = \frac{Cs[V(s) + Li(0)]}{CLs^2 + CRs + 1},$$

since $I(0)=0$. Given the data $L=2$, $R=3$, $C=\frac{1}{3}$ and $v(t)=3\cos t \leftrightarrow 3s/(s^2+1)$:

$$\begin{aligned} I(s) &= \frac{3s^2}{(s^2+1)(2s^2+3s+3)} = \frac{3(3s-1)}{10(s^2+1)} - \frac{9(2s-1)}{10(2s^2+3s+3)} \\ &= \frac{3(3s-1)}{10(s^2+1)} - \frac{9(2s-1)}{20[(s+\frac{3}{4})^2+\frac{15}{16}]} \end{aligned}$$

Using table (24.10) and the shift rule (24.7), the inverse is

$$i(t) = \frac{9}{10}\cos t - \frac{3}{10}\sin t + \frac{3\sqrt{15}}{10}\mathrm{e}^{-\frac{3}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right) - \frac{9}{10}\mathrm{e}^{-\frac{3}{4}t}\cos\left(\frac{\sqrt{15}}{4}t\right).$$

(b) If $v(t)=0$ and the capacitor has initial charge q_0, the equation for the current is (see Section 25.1)

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + \frac{1}{C}\left(\int_0^t i(\tau)\mathrm{d}\tau + q_0\right) = 0.$$

Taking the Laplace transform of the equation, noting that $i(0)=0$,

$$LsI(s) + RI(s) + \frac{1}{Cs}I(s) + \frac{q_0}{Cs} = 0.$$

Hence

$$I(s) = -\frac{q_0}{CLs^2+CRs+1} = -\frac{3q_0}{2s^2+3s+3} = -\frac{3q_0}{2[(s+\frac{3}{4})^2+\frac{15}{16}]}.$$

The inverse of this transform is

$$i(t) = -2\sqrt{\frac{3}{5}}q_0\mathrm{e}^{-\frac{3}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right).$$

(c) We can represent the applied voltage by $v(t) = 300\times 0.01\delta(t-t_0) = 3\delta(t-t_0)$. Hence the circuit equation for the current is

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + \frac{1}{C}\int_0^t i(\tau)\mathrm{d}\tau = 3\delta(t-t_0).$$

Hence

$$I(s) = \frac{3s\mathrm{e}^{-st_0}}{2s^2+3s+3} = \frac{3s\mathrm{e}^{-st_0}}{2[(s+\frac{3}{4})^2+\frac{15}{16}]}.$$

Using the shift rule (24.7) and the delay rule (24.15), the inverse of the Laplace transform $I(s)$ is

$$i(t) = \mathrm{e}^{-\frac{3}{4}(t-t_0)}\left[-\frac{3}{2}\cos\left(\frac{\sqrt{15}}{4}(t-t_0)\right) + \frac{3\sqrt{15}}{10}\sin\left(\frac{\sqrt{15}}{4}(t-t_0)\right)\right]\mathrm{H}(t-t_0).$$

25.3. The equation of motion is

$$\ddot{x} + 2k\dot{x} + \omega^2 x = f(t).$$

The impulse can be represented by a delta function so that $f(t) = I\delta(t-t_0)$. Take the Laplace transform of the equation, noting that $x(0)=1$ and $\dot{x}(0)=1$,

$$s^2X(s) - s - 1 + 2ksX(s) - 2k + \omega^2X(s) = \mathcal{L}\{I\delta(t-t_0)\} = I\mathrm{e}^{-t_0s}.$$

Hence

$$X(s) = \frac{s+1+2k+I\mathrm{e}^{-t_0s}}{s^2+2ks+\omega^2} = \frac{s+1+2k+I\mathrm{e}^{-t_0s}}{(s+k)^2-\beta^2}, \qquad \beta^2 = k^2-\omega^2.$$

The term containing $e^{-t_0 s}$ arising from the impulse will lead to the term with a step function on inversion (see the delay rule (24.15)). The full solution is

$$\begin{aligned} x &= \frac{1}{2\beta}[(-k-1+\beta)e^{-t(k+\beta)} + (k+1+\beta)e^{-t(k-\beta)}] \\ &\quad + \frac{I}{2\beta}[e^{-(t-t_0)(k-\beta)} - e^{-(t-t_0)(k+\beta)}]H(t-t_0). \end{aligned}$$

for $t > 0$ and $t_0 > 0$.

25.4. The displacement $u(x)$ of the plank satisfies

$$K\frac{d^4u}{dx^4} = f(x).$$

The mountaineer standing at the centre of the plank is treated as a point load which can be represented by the delta function $Mg\delta(x - \frac{1}{2}l)$, so that

$$K\frac{d^4u}{dx^4} = Mg\delta(x - \frac{1}{2}l).$$

Let $A = u'(0)$ and $B = u'''(0)$ since only $u(0) = 0$ and $u''(0) = 0$ are given: the constants A and B will be found from the conditions at $x = l$ when we have solved the equation. The Laplace transform of the equation is

$$Ks^4U(s) - As^2 - B = \mathcal{L}\{Mg\delta(x - \tfrac{1}{2}l)\} = Mge^{-\frac{1}{2}ls}.$$

Therefore

$$U(s) = \frac{1}{K}\left[\frac{A}{s^2} + \frac{B}{s^4} + Mg\frac{e^{-\frac{1}{2}ls}}{s^4}\right].$$

Inversion of this transform using table (24.10) and delay rule (24.15) gives

$$u(x) = \frac{1}{K}\left[Ax + \tfrac{1}{6}Bx^3 + \tfrac{1}{6}Mg(x - \tfrac{1}{2}l)^3H(x - \tfrac{1}{2}l)\right].$$

The conditions at $x = l$ are $u(l) = 0$ and $u''(l) = 0$. Hence

$$u(l) = \frac{1}{K}\left[Al + \tfrac{1}{6}Bl^3 + \tfrac{1}{6}Mg(l - \tfrac{1}{2}l)^3H(\tfrac{1}{2}l)\right] = \frac{1}{K}\left[Al + \tfrac{1}{6}Bl^3 + \tfrac{1}{48}Mgl^3\right] = 0,$$

and

$$u''(l) = \frac{1}{K}\frac{d^2}{dx^2}\left[Ax + \tfrac{1}{6}Bx^3 + \tfrac{1}{6}Mg(x - \tfrac{1}{2}l)^3\right]_{x=l} = \frac{1}{K}\left[Bl + \tfrac{1}{2}Mgl\right] = 0.$$

Solving these equations

$$B = -\tfrac{1}{2}Mg, \quad A = \tfrac{1}{16}Mgl^2.$$

25.5. Use the impedance rules listed in (25.8) and (25.9).
(a) For the resistor r and inductor in parallel the impedance Z_1 is given by

$$\frac{1}{Z_1} = \frac{1}{R} + \frac{1}{Ls}.$$

Therefore

$$Z_1 = \frac{RLs}{R + Ls}.$$

The impedance Z_1 is in series with the capacitor C. If Z is the impedance of the whole circuit, then

$$Z = Z_1 + \frac{1}{Cs} = \frac{RLs}{R + lsL} + \frac{1}{Cs}.$$

Applying the given data

$$Z = \frac{6s}{2+3s} + \frac{2}{s}.$$

(b) These components are all in parallel. Hence the impedance Z is given by

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{Ls} + \frac{1}{1/(Cs)} = \frac{Ls + R + CRLs^2}{RLs}.$$

Hence

$$Z = \frac{RLs}{CRLs^2 + Ls + R} = \frac{2s}{6s^2 + s + 1}$$

for the given data.

(c) For the parallel resistor $R = 1$ and the inductor $L = 1$ the impedance Z_1 is given by

$$\frac{1}{Z_1} = 1 + \frac{1}{s} \text{ so that } Z_1 = \frac{s}{s+1}.$$

The impedance is in series with the resistor $R = 2$ which has the impedance

$$Z_2 = 2 + Z_1 = 2 + \frac{s}{s+1} = \frac{3s+2}{s+1}.$$

Finally Z_2 is in parallel with the capacitor $C = 2$ giving the impedance

$$\frac{1}{Z} = \frac{1}{Z_2} + \frac{1}{2/s} = \frac{6s^2 + 5s + 1}{2(3s+2)}.$$

Therefore

$$Z = \frac{3s+2}{6s^2 + 5s + 1}.$$

25.6. (a) Let $I_1(s)$ be the current in the s domain through the capacitor $C = 2$, $I_2(s)$ be the current through the resistor $R = 3$ and the inductor $L = 1$ and $I_3(s)$ through the resistor $R = 5$. Then by Kirchhoff's laws

$$I(s) - I_1(s) - I_2(s) = 0, \quad I_1(s) + I_2(s) - I_3(s) = 0,$$

and

$$V_1(s) = I_2(s)(3 + s) + 5I_3(s),$$

$$\frac{I_1(s)}{2s} - I_2(s)(s+3) = 0, \quad V_2(s) = \frac{I_1(s)}{2s}.$$

It follows from the first two equation that $I_2(s) = I(s) - I_1(s)$ and that (not surprisingly) that $I_3(s) = I_1(s) + I_2(s) = I(s)$. Hence

$$V_1(s) = (I(s) - I_1(s))(s+3) + 5I(s),$$

and

$$\frac{I_1(s)}{2s} - (I(s) - I_1(s))(s+3) = 0.$$

Eliminate $I(s)$ between these equations so that

$$V_1(s) = 2I_1(s)(2s^2 + 5s - 2)).$$

Therefore

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{2s(2s^2 + 5s - 2)}.$$

and

$$\frac{V_2(s)}{I(s)} = \frac{1}{2s(2s^2 + 6s + 1)}.$$

(b) Let $I_1(s)$ be the current in the s domain through the inductor $L = 2$. Apply Kirchhoff's law in the s domain to the three subcircuits in Figure 25.22(b). Then

$$V_1(s) = 2(I_1(s) + I(s)) + sI(s) = 2I_1(s) + 5I(s), \tag{i}$$

$$\left(2s + \frac{1}{2s}\right) I_1(s) - 3I(s) = 0, \tag{ii}$$

$$V_2(s) = \frac{I_1(s)}{2s}. \tag{iii}$$

From (ii) $I(s) = I_1(s)(4s^2 + 1)/(6s)$. Using this equation eliminate $I(s)$ in (i) so that

$$V_1 = \frac{(20s^2 + 12s + 5)I_1(s)}{6s}.$$

Finally combining this equation with (iii):

$$\frac{V_2(s)}{V_1(s)} = \frac{3s}{20s^2 + 12s + 5}.$$

and

$$\frac{V_2(s)}{I(s)} = \frac{I_1(s)}{2s}\frac{6s}{I_1(s)(4s^2+1)} = \frac{3}{4s^2+1}.$$

25.7. The convolution theorem (25.11) states that if

$$f(t) = \int_0^t g(\tau)h(t-\tau)\mathrm{d}\tau = \int_0^t h(\tau)g(t-\tau)\mathrm{d}\tau, \quad G(s) \leftrightarrow g(t), \quad H(s) \leftrightarrow h(t),$$

then

$$F(s) = \mathcal{L}\{f(t)\} = G(s)H(s).$$

The convolution integrals are either integrated directly or by using the convolution theorem.
(a) $g(t) = \mathrm{e}^t$, $h(t) = 1$. By direct integration

$$f(t) = \int_0^t g(\tau)h(t-\tau)\mathrm{d}\tau = \int_0^t \mathrm{e}^\tau \mathrm{d}\tau = [\mathrm{e}^\tau]_0^t = \mathrm{e}^t - 1.$$

Alternatively, using the convolution theorem and $g(t) = \mathrm{e}^t \leftrightarrow 1/(s-1) = G(s)$, $h(t) = 1 \leftrightarrow 1/s = H(s)$, we obtain

$$F(s) = G(s)H(s) = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s} \leftrightarrow \mathrm{e}^t - 1 = f(t).$$

(b) $g(t) = 1$, $h(t) = 1$. By direct integration

$$f(t) = \int_0^t g(\tau)h(t-\tau)\mathrm{d}\tau = \int_0^t \mathrm{d}\tau = t.$$

(c) $g(t) = \mathrm{e}^t$, $h(t) = \mathrm{e}^t$. By direct integration

$$f(t) = \int_0^t g(\tau)h(t-\tau)\mathrm{d}\tau = \int_0^t \mathrm{e}^\tau \mathrm{e}^{t-\tau}\mathrm{d}\tau = \mathrm{e}^t \int_0^t \mathrm{d}\tau = t\mathrm{e}^t.$$

(d) $g(t) = \mathrm{e}^{-t}$, $h(t) = t$. Since $\mathrm{e}^{-t} \leftrightarrow 1/(s+1)$ and $t \leftrightarrow 1/s^2$, the convolution theorem gives

$$F(s) = G(s)H(s) = \frac{1}{s^2(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} \leftrightarrow -1 + t + \mathrm{e}^{-t} = f(t).$$

(e) $g(t) = t$, $h(t) = \sin t$. Using the tables of transforms, $t \leftrightarrow 1/s^2$ and $\sin t \leftrightarrow 1/(s^2+1)$, the convolution of $g(t)$ and $h(t)$ has the transform

$$F(s) = G(s)H(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} \leftrightarrow t - \sin t = f(t).$$

(f) $g(t) = \cos t$, $h(t) = t$. Using tables of transforms, $t \leftrightarrow 1/s^2$ and $\cos t \leftrightarrow s/(s^2+1)$, the convolution of the transform is

$$F(s) = G(s)H(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \leftrightarrow 1 - \cos t = f(t).$$

(g) $g(t) = \sin 3t$, $h(t) = \mathrm{e}^{-2t}$. The transforms of $g(t)$ and $h(t)$ are $\sin 3t \leftrightarrow 3/(s^2+9)$ and $\mathrm{e}^{-2t} \leftrightarrow 1/(s+2)$. Hence by the convolution theorem

$$\begin{aligned} F(s) &= G(s)H(s) = \frac{3}{(s+2)(s^2+9)} \\ &= \frac{3}{13(s+2)} + \frac{6}{13(s^2+9)} - \frac{3s}{13(s^2+9)} \\ &\leftrightarrow \frac{3}{13}\mathrm{e}^{-2t} + \frac{2}{13}\sin 3t - \frac{3}{13}\cos 3t = f(t) \end{aligned}$$

(h) $g(t) = h(t) = \sin t$. Since $\sin t \leftrightarrow 1/(s^2+1)$, the convolution theorem gives

$$\begin{aligned} F(s) &= G(s)H(s) = \frac{1}{(s^2+1)^2} = \frac{1}{2}\left[-\frac{s^2-1}{(s^2+1)^2} + \frac{1}{s^2+1}\right] \\ &\leftrightarrow -\frac{1}{2}t\cos t + \frac{1}{2}\sin t = f(t) \end{aligned}$$

by (24.9).

(i) $g(t) = t^4$, $h(t) = \sin t$. Since $t^4 \leftrightarrow 24/s^5$ and $\sin t \leftrightarrow 1/(s^2+1)$, the convolution theorem gives

$$\begin{aligned} F(s) &= G(s)H(s) = \frac{24}{s^5(s^2+1)} = 24\left[\frac{1}{s^5} - \frac{1}{s^3} + \frac{1}{s} - \frac{s}{s^2+1}\right] \\ &\leftrightarrow t^4 - 12t^2 + 24 - 24\cos t = f(t) \end{aligned}$$

(j) $g(t) = t^n$, $h(t) = t^m$. Since $t^n \leftrightarrow n!/s^{n+1}$ and $t^m \leftrightarrow m!/s^{m+1}$ (assuming that n and m are positive integers), the convolution theorem gives

$$F(s) = G(s)H(s) = \frac{n!m!}{s^{n+m+2}} \leftrightarrow \frac{n!m!t^{n+m+1}}{(n+m+1)!} = f(t).$$

25.8. (a) The Laplace transform of

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = f(t)$$

is

$$s^2X(s) - sx(0) - x'(0) + \omega^2X(s) = F(s),$$

where $X(s) = \mathcal{L}\{x(t)\}$ and $F(s) = \mathcal{L}\{f(t)\}$. Choose a particular solution such that $x(0) = 0$ and $x'(0) = 0$. Then

$$X(s) = \frac{F(s)}{s^2+\omega^2}.$$

From (24.6)

$$\frac{1}{s^2+\omega^2} \leftrightarrow \sin t.$$

Hence by the convolution theorem, a particular solution is

$$x(t) = \int_0^t f(\tau)\sin(t-\tau)\mathrm{d}\tau.$$

(b) The Laplace transform of

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \omega^2 x = f(t)$$

is.

$$s^2X(s) - sx(0) - x'(0) - \omega^2 X(s) = F(s).$$

Choose a particular solution such that $x(0) = 0$ and $x'(0) = 0$. Then

$$X(s) = \frac{F(s)}{s^2-\omega^2} = \frac{1}{2\omega}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] \leftrightarrow \frac{1}{2\omega}[\mathrm{e}^{\omega t} - \mathrm{e}^{-\omega t}].$$

Hence by the convolution theorem, a particular solution is

$$x(t) = \frac{1}{2\omega}\int_0^t f(\tau)[\mathrm{e}^{\omega(t-\tau)} - \mathrm{e}^{-\omega(t-\tau)}]\mathrm{d}\tau.$$

25.9. (a) $\int_0^t x(\tau)(t-\tau)\mathrm{d}\tau = t^4$. Let $X(s) = \mathcal{L}\{x(t)\}$. The transform of the equation is

$$X(s)\mathcal{L}\{t\} = \mathcal{L}\{t^4\}, \text{ or } X(s)\frac{1}{s^2} = \frac{4!}{s^5}.$$

Hence

$$X(s) = \frac{24}{s^3} = 12t^2.$$

Therefore the solution of the integral equation is $x(t) = 12t^2$.
(b) $x(t) = 1 + \int_0^t x(\tau)(t-\tau)\mathrm{d}\tau$. Take the Laplace transform of the equation using the convolution theorem:

$$X(s) = \frac{1}{s} + \frac{X(s)}{s^2} \text{ so that } X(s) = \frac{s}{s^2-1} = \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+1}\right],$$

using partial fractions. Inversion gives the solution

$$x(t) = \tfrac{1}{2}[\mathrm{e}^{-t} + \mathrm{e}^{t}] = \cosh t.$$

(c) $x(t) = \sin t + \int_0^t x(\tau)\cos(t-\tau)\mathrm{d}\tau$. The Laplace transform of this integral equation is

$$X(s) = \frac{1}{s^2+1} + X(s)\frac{s}{s^2+1},$$

using the convolution theorem. Hence

$$X(s) = \frac{1}{s^2-s+1} = \frac{1}{(s-\frac{1}{2})^2 + \frac{3}{4}}.$$

We can invert this transform using (24.6c) and the shift rule (24.7) resulting in the solution

$$x(t) = \tfrac{2}{\sqrt{3}}\mathrm{e}^{\frac{1}{2}t}\sin[\tfrac{1}{2}\sqrt{3}t].$$

25.10. The input $f(t) = \mathrm{H}(t)$ has the transform $F(s) = 1/s$. As in Section 25.6, the response $x^{**}(t)$ has the transform

$$X^{**}(s) = F(s)G(s) = \frac{G(s)}{s} \text{ so that } G(s) = sX^{**}(s),$$

where $G(s)$ is the transfer function between input and output. The transform of the output $x(t)$ is

$$X(s) = G(s)F(s) = X^{**}(s)sF(s).$$

Using the convolution theorem, the output

$$x(t) = \int_0^t x^{**}(t-\tau)\left[\frac{\mathrm{d}}{\mathrm{d}\tau}(f(\tau)) + f(0)\right]\mathrm{d}\tau,$$

by the derivative rule (24.12). Let

$$X^{**}(s)F(s) \leftrightarrow q(t) = \int_0^t x^{**}(\tau)f(t-\tau)\mathrm{d}\tau.$$

By the rule on the differentiation of integrals (15.20), the right-hand side can be expressed as

$$x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t x^{**}(\tau)f(t-\tau)\mathrm{d}\tau\right].$$

In the example

$$X^{**}(s) = \frac{1}{(s-1)(s+2)} = \frac{1}{3(s-1)} - \frac{1}{3(s+2)} \leftrightarrow \frac{1}{3}(\mathrm{e}^t - \mathrm{e}^{-2t}),$$

and $f(t) = \mathrm{H}(t)\sin\omega t$.

25.11. To model this problem, approximate by assuming that the learning/ forgetting process takes place continuously through whatever time-range is involved. Note that the learning data refers to *new* words (forgotten words are not revised). The words learned at a time $t = \tau$ through a small period $\delta\tau$ is equal to $50\delta\tau$. at any later time t, the number of newly-learned words that are remembered is $50\delta\tau\mathrm{e}^{-0.01(t-\tau)}$, (that is, the elapsed time α is $t-\tau$ for this event). The total number of words recalled at time t is the limit of the sum of the contributions between times 0 and t:

$$N(t) = \lim_{\delta\tau\to 0}\sum_{\tau=0}^{t} 50\mathrm{e}^{-0.01(t-\tau)}\delta\tau = \int_0^t 50\mathrm{e}^{-0.01(t-\tau)}\mathrm{d}\tau \qquad \text{(i)}$$

(which is the convolution $50 * \mathrm{e}^{-0.01t}$). Therefore

$$N(t) = 50\mathrm{e}^{-0.01t}\left[\frac{\mathrm{e}^{0.01\tau}}{0.01}\right]_0^t = 50\frac{1-\mathrm{e}^{-0.01t}}{0.01}.$$

(After 30 days $N(t)$ becomes 50×25.9 in place of 50×30 attempted: a loss of 14%.)

(b) If the student aims at learning $50 + 0.1t$ words per day (thus increasing the input with time) we obtain the convolution integral

$$N(t) = \int_0^t (50+0.1\tau)\mathrm{e}^{-0.01(t-\tau)}\mathrm{d}\tau = \mathrm{e}^{-0.01t}\int_0^t (50+0.1\tau)\mathrm{e}^{0.01\tau}\mathrm{d}\tau \qquad \text{(ii)}$$

(by 25.11). The integration by parts formula (17.8) with $u = \tau$, $\mathrm{d}v/\mathrm{d}\tau = \mathrm{e}^{A\tau}$ gives

$$\int_0^t \tau\mathrm{e}^{A\tau}\mathrm{d}\tau = \left[\frac{1}{A}\left(\tau - \frac{1}{A}\right)\mathrm{e}^{A\tau}\right]_0^t = \frac{1}{A}\left[t\mathrm{e}^{At} + \frac{1}{A}(1-\mathrm{e}^{At})\right],$$

and applying this to (ii) with $A = 0.01$,

$$\begin{aligned} N(t) &= \mathrm{e}^{-0.01t}\left[-50\frac{1-\mathrm{e}^{0.01t}}{0.01} + \frac{0.1}{0.01}\left(t\mathrm{e}^{0.01t} + \frac{1-\mathrm{e}^{0.01t}}{0.01}\right)\right] \\ &= \mathrm{e}^{-0.01t}\left[-40\frac{1-\mathrm{e}^{0.01t}}{0.01} + 10t\mathrm{e}^{0.01t}\right]. \end{aligned}$$

25.12. The original population p_0 declines as $p_0\mathrm{e}^{-\gamma t}$. In time $\delta\tau$, the number born is $bp(\tau)\delta\tau$, but these individuals die out at the rate $\mathrm{e}^{-\beta(t-\tau)}$ after the elapsed time $t-\tau$. At time t the balance between these births and deaths together with the decline of p_0 gives the population $p(t)$ at time t:

$$p(t) = p_0\mathrm{e}^{-\gamma t} + b\int_0^t p(\tau)\mathrm{e}^{-\beta(t-\tau)}\mathrm{d}\tau.$$

Taking the Laplace transform of this equation, which includes a convolution on the right-hand side, we obtain

$$P(s) = \frac{p_0}{s+\gamma} + \frac{bP(s)}{s+\beta},$$

where $P(s) = \mathcal{L}\{p(t)\}$. Therefore

$$P(s) = \frac{p_0(s+\beta)}{(s+\gamma)(s+\beta-b)} = \frac{p_0}{b+\gamma-\beta}\left[\frac{b}{s+\beta-b} + \frac{\gamma-\beta}{s+\gamma}\right].$$

This transform can now be inverted using (24.6a) giving the solution

$$p(t) = \frac{p_0}{b+\gamma-\beta}\left[b\mathrm{e}^{(b-\beta)t} + (\gamma-\beta)\mathrm{e}^{-\gamma t}\right].$$

25.13. The equation of motion is

$$m\ddot{x} + kx = F_0[\mathrm{H}(t) - \mathrm{H}(t-t_0)],$$

where the difference of the unit functions on the right ensures that the forcing is zero for $t > t_0$. The initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$. The Laplace transform of this equation is

$$ms^2X(s) + kX(s) = F_0\left[\frac{1}{s} - \frac{\mathrm{e}^{t_0 s}}{s}\right].$$

Therefore

$$X(s) = \frac{F_0(1-\mathrm{e}^{-t_0 s})}{ms(s^2+\omega^2)} = \frac{F_0}{m\omega^2}\left[\frac{1-\mathrm{e}^{-t_0 s}}{s} - \frac{s(1-\mathrm{e}^{-t_0 s})}{s^2+\omega^2}\right]$$

where $\omega^2 = k/m$. Inversion gives

$$x(t) = \frac{F_0}{k}[(1-\cos\omega t) - (1-\cos\omega(t-t_0))\mathrm{H}(t-t_0)].$$

Hence for $t < t_0$, the solution is

$$x(t) = \frac{F_0}{k}(1-\cos\omega t),$$

whilst for $t > t_0$ the solution is

$$x(t) = \frac{F_0}{k}[\cos\omega(t-t_0) - \cos\omega t].$$

25.14. The differential equation is

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t-1) + t,$$

and $x(t) = 0$ for $t \le 0$. Take the Laplace transform of the equation:

$$sX(s) - x(0) = \frac{1}{s^2} + \int_0^\infty x(t-1)\mathrm{e}^{-st}\mathrm{d}t,$$

where $X(s) = \mathcal{L}\{x(t)\}$. Since $x(t) = 0$ for $t \le 0$, and by using the second shift rule (24.15),

$$sX(s) = \frac{1}{s^2} + \int_1^\infty x(t-1)\mathrm{e}^{-st}\mathrm{d}t = \frac{1}{s^2} + \int_0^\infty x(\tau)\mathrm{e}^{-s(u+1)}\mathrm{d}u = \frac{1}{s^2} + \mathrm{e}^{-s}X(s).$$

Hence
$$X(s) = \frac{1}{s^2(s-\mathrm{e}^{-s})} = \frac{1}{s^3(1-\mathrm{e}^{-s}/s)}.$$
Using the binomial expansion
$$X(s) = \frac{1}{s^3}\left(1-\frac{\mathrm{e}^{-s}}{s}\right)^{-1} = \frac{1}{s^3} + \frac{\mathrm{e}^{-s}}{s^4} + \frac{\mathrm{e}^{-2s}}{s^5} + \cdots.$$
The general term in this series is e^{-ns}/s^{n+3}. The function of which this the Laplace transform is
$$\frac{(t-n)^{n+2}}{(n+2)!}\mathrm{H}(t-n) = \begin{cases} (t-n)^{n+2}/(n+2)! & n \le t \\ 0 & n > 0 \end{cases}$$
We sum the series as far as n where $\lfloor t-1 \rfloor < n \le \lfloor t \rfloor$ and $\lfloor t \rfloor$ is the largest integer less than or equal to t. Therefore
$$x(t) = \sum_{n=0}^{\lfloor t \rfloor} \frac{(t-n)^{n+2}}{(n+2)!}.$$

25.15. The Laplace transform of the integral equation
$$2\int_0^t \cos(t-u)x(u)\mathrm{d}u = x(t) - t$$
is
$$2\frac{s}{s^2+1}X(s) = X(s) - \frac{1}{s^2}.$$
Hence
$$X(s) = \frac{s^2+1}{s^2(s-1)^2} = \frac{2}{(s-1)^2} - \frac{2}{s-1} + \frac{1}{s^2} + \frac{2}{s}$$
using partial fractions. Finally, inverting this transform, the required solution is
$$x(t) = 2(t-1)\mathrm{e}^t + t + 2.$$

25.16. The differential equation is
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + t\frac{\mathrm{d}x}{\mathrm{d}t} - x = 0,$$
where $x(0) = 0$ and $x'(0) = 1$. The derivative $\mathrm{d}x/\mathrm{d}t$ has a variable coefficient t. From (24.8)
$$\mathcal{L}\left\{t\frac{\mathrm{d}x}{\mathrm{d}t}\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left[sX(s) - x(0)\right] = X(s) + s\frac{\mathrm{d}X(s)}{\mathrm{d}s}.$$

Hence the Laplace transform of the full equation is
$$s^2X(s) - 1 - X(s) - s\frac{\mathrm{d}X(s)}{\mathrm{d}s} - X(s) = 0.$$
Therefore $X(s)$ satisfies
$$-s\frac{\mathrm{d}X(s)}{\mathrm{d}s} + (s^2-2)X(s) = 1.$$
Let $X(s) = 1/s^2$. Then
$$-s\frac{\mathrm{d}X(s)}{\mathrm{d}s} + (s^2-2)X(s) = \frac{2}{s^2} + (s^2-2)\frac{1}{s^2} = 1.$$
Hence $X(s) = 1/s^2$ satisfies the equation. The corresponding time-solution is $x(t) = t$ which we can confirm satisfies the initial conditions.

25.17. These equations have some coefficients which are not constant, and their transforms are obtained by using the multiplication rule (24.8).

(a) $tx''(t) + (1-t)x'(t) - x(t) = 0$, $x(0) = x'(0) = 1$. Take the Laplace transform of the equation:

$$-\frac{\mathrm{d}}{\mathrm{d}s}[s^2X(s) - sx(0) - x'(0)] + sX(s) - x(0) + \frac{\mathrm{d}}{\mathrm{d}s}[sX(s) - x(0)] - X(s) = 0,$$

or

$$-2sX(s) - s^2X'(s) + 1 + sX(s) - 1 + X(s) + sX'(s) - X(s) = 0.$$

Hence

$$(s-1)X'(s) + X(s) = 0.$$

This a first-order separable equation with general solution $X(s) = C/(s-1)$, where C is a constant. By the Table of Laplace transforms (Appendix F), inversion gives $x(t) = C\mathrm{e}^t$. The initial condition $x(0) = 1$ means that $C = 1$. Hence the required solution is $x(t) = \mathrm{e}^t$.

(b) $x''(t) + tx'(t) - 2x(t) = 2$, $x(0) = x'(0) = 0$. Take the Laplace transform of the equation:

$$s^2X(s) - sx(0) - x'(0) - \frac{\mathrm{d}}{\mathrm{d}s}[sX(s) - x(0)] - 2X(s) = \frac{2}{s},$$

or

$$s^2X(s) - X(s) - sX'(s) - 2X(s) = \frac{2}{s}.$$

Hence

$$X'(s) - \left(s - \frac{3}{s}\right)X(s) = -\frac{2}{s^2}.$$

This is a first-order differential equation of integrating-factor type (see Section 19.5). The integrating factor is

$$\mathrm{e}^{\int(-s+3/s)\mathrm{d}s} = \mathrm{e}^{(-\frac{1}{2}s^2+3\ln s)} = s^3\mathrm{e}^{-\frac{1}{2}s^2}.$$

The equation can be expressed in the form

$$\frac{\mathrm{d}}{\mathrm{d}s}(X(s)s^3\mathrm{e}^{-\frac{1}{2}s^2}) = -2s\mathrm{e}^{-\frac{1}{2}s^2}.$$

Integrating

$$X(s)s^3\mathrm{e}^{-\frac{1}{2}s^2} = -\int 2s\mathrm{e}^{-\frac{1}{2}s^2}\mathrm{d}s + C = 2\mathrm{e}^{-\frac{1}{2}s^2} + C,$$

where C is a constant. Hence the transform of the solution is

$$X(s) = \frac{2}{s^3} + C\frac{\mathrm{e}^{\frac{1}{2}s^2}}{s^3}.$$

Do not attempt to invert the second term on the right. The inverse of the first term is t^2. This term alone satisfies the initial conditions $x(0) = x'(0) = 0$. We conclude that $C = 0$ which means that the required solution is $x(t) = t^2$.

(c) $tx''(t) - x'(t) + tx(t) = \sin t$, $x(0) = 1$, $x'(0) = 0$. The Laplace transform of the equation is

$$-\frac{\mathrm{d}}{\mathrm{d}}[s^2X(s) - sx(0) - x'(0)] - sX(s) + x(0) - \frac{\mathrm{d}X(s)}{\mathrm{d}s} = \frac{1}{s^2+1},$$

or

$$X'(s) + \frac{3s}{s^2+1}X(s) = \frac{2}{s^2+1} - \frac{1}{(s^2+1)^2}.$$

This is an equation of integrating-factor type as in (b). The integrating factor is

$$\mathrm{e}^{\int 3s/(s^2+1)\mathrm{d}s} = \mathrm{e}^{\frac{3}{2}\ln(s^2+1)} = (s^2+1)^{\frac{3}{2}},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}s}[X(s)(s^2+1)^{\frac{3}{2}}] = 2(s^2+1)^{\frac{1}{2}} - (s^2+1)^{-\frac{1}{2}}.$$

Integrating:

$$\begin{aligned} X(s)(s^2+1)^{\frac{3}{2}} &= 2\int (s^2+1)^{\frac{1}{2}}\mathrm{d}s - \int (s^2+1)^{-\frac{1}{2}}\mathrm{d}s + C \\ &= \int \frac{\mathrm{d}}{\mathrm{d}s}[s(s^2+1)^{\frac{1}{2}}]\mathrm{d}s + C \\ &= s(s^2+1)^{\frac{1}{2}} + C \end{aligned}$$

Hence

$$X(s) = \frac{s}{s^2+1} + \frac{C}{(s^2+1)^{\frac{3}{2}}}.$$

The inverse of the first term is $\cos t$ which alone satisfies the initial conditions. Therefore $C = 0$ and the required solution is $x(t) = \cos t$.

25.18. (a) $\{1, 2, 1, 0, 0, 0, \ldots\}$, $1+\frac{2}{z}+\frac{1}{z}$, and the Laplace transfrom $1+2\mathrm{e}^{-Ts}+\mathrm{e}^{-2Ts}$ are equivalent. They represent the sum of the impulses: $\delta(t) + 2\delta(t-T) + \delta(t-2T)$.

(b) The sequence $\{0, 1, 2, 3, \ldots\}$; the z transform

$$\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots,$$

and the Laplace transform

$$\mathrm{e}^{-Ts} + 2\mathrm{e}^{-2Ts} + 3\mathrm{e}^{-3Ts} + \cdots$$

are equivalent, representing the time function

$$\delta(t-T) + 2\delta(t-2T) + 3\delta(t-3T) + \cdots.$$

Note also that

$$\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots = \frac{z}{(1-z)^2}$$

for large z (compare Example 25.18), which is the z transform in finite terms.

(c) The notation $\{3\}$ means here the sequence $\{3, 0, 0, 0, \ldots\}$, standing for the z transform equal to $3/z^0 = 3$. The Laplace transform therefore equals 3, corresponding to the time function $3\delta(t)$.

(d) $\{(-2)^n\}$, $\{1, -2, 2^2, -2^3, \ldots\}$, $1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots$, and
$1 - 2\mathrm{e}^{-Ts} + 2^2\mathrm{e}^{-2Ts} - \cdots$ are all equivalent, and correspond to the time function $\delta(t) - 2\delta(t-T) + 2^2\delta(t-2T) - \cdots$. In finite terms: the (geometric) series for the z transform has the sum

$$\frac{1}{1+2/z} = \frac{z}{z+2}$$

for large z.

(e) $\{0, 0, 3\}$, or $3/z^2$, is the z transform, $3\mathrm{e}^{-2Ts}$ the Laplace transform and $3\delta(t-2T)$ the time function.

25.19. *Note.* The periods are denoted by T. Inputs and outputs are related through their z transforms by $\mathcal{Y}(z) = \mathcal{G}(z)\mathcal{X}(z)$ (see (25.26)).

(a)

$$\mathcal{Y}(z) = \left(1+\frac{1}{z}\right)\left(1+\frac{1}{z}\right) = 1 + \frac{2}{z} + \frac{1}{z^2}.$$

The inverse is

$$y(t) = \delta(t) + 2\delta(t-T) + \delta(t-2T).$$

(b)

$$\mathcal{G}(z) = 1 + \frac{1}{2z} + \frac{1}{2^2z^2} + \cdots, \text{ and } \mathcal{X}(z) = 1 + \frac{1}{z}.$$

Therefore

$$\begin{aligned}\mathcal{Y}(z) &= 1 + \left(\frac{1}{2}+1\right)\frac{1}{z} + \left(\frac{1}{2^2}+\frac{1}{2}\right)\frac{1}{z^2} + \left(\frac{1}{2^3}+\frac{1}{2^2}\right)\frac{1}{z^3} + \cdots \\ &= 1 + \frac{3}{2}\left(\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{2^2z^3} + \cdots\right).\end{aligned}$$

The inverse is given by

$$\mathcal{Y}(t) = \delta(t) + \frac{3}{2}\left[\delta(t-T) + \frac{1}{2}\delta(t-2T) + \frac{1}{2^2}\delta(t-3T) + \cdots\right].$$

(c)

$$\begin{aligned}\mathcal{Y}(z) &= \left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right)\left(\frac{2}{z} + \frac{2}{z^2}\right) = \frac{2}{z}\left(1+\frac{1}{z}\right)\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right) \\ &= \frac{2}{z}\left(1+\frac{1}{z}\right)\cdot\frac{1}{1+\frac{1}{z}} = \frac{2}{z}.\end{aligned}$$

Therefore $y(t) = 2\delta(t-T)$.

25.20. $G(s) = 1/(1 - \frac{1}{3}\mathrm{e}^{-Ts})$, $X(s) = \mathrm{e}^{-Ts} + 2\mathrm{e}^{-2Ts}$. Put $\mathrm{e}^{Ts} = z$ (see (25.22)); then the corresponding z transforms are given by

$$\mathcal{G} = \frac{1}{1 - \frac{1}{3}\frac{1}{z}}, \quad \mathcal{X}(z) = \frac{1}{z} + \frac{2}{z^2}.$$

Expand $\mathcal{G}(z)$ as a geometric series (see (5.4a)) in powers of $1/z$:

$$\mathcal{G}(z) = 1 + \frac{1}{3}\frac{1}{z} + \frac{1}{3^2}\frac{1}{z^2} + \cdots.$$

Since $\mathcal{Y}(z) = \mathcal{G}(z)\mathcal{X}(z)$ (see (25.26)), we have

$$\begin{aligned}\mathcal{Y}(z) &= \left(1 + \frac{1}{3}\frac{1}{z} + \frac{1}{3^2}\frac{1}{z^2} + \cdots\right)\frac{1}{z}\left(1 + \frac{2}{z}\right) \\ &= \frac{1}{z}\left(1 + \left(\frac{1}{3}+2\right)\frac{1}{z} + \frac{1}{3}\left(\frac{1}{3}+2\right)\frac{1}{z^2} + \frac{1}{3^2}\left(\frac{1}{3}+2\right)\frac{1}{z^3} + \cdots\right) \\ &= \frac{1}{z} + \frac{7}{3}\left(\frac{1}{z^2} + \frac{1}{3z^3} + \frac{1}{3^2z^4} + \cdots\right)\end{aligned}$$

The output $y(t)$ is therefore given by

$$y(t) = \delta(t-T) + \tfrac{7}{3}[\delta(t-2T) + \tfrac{1}{3}\delta(t-3T) + \tfrac{1}{3^2}\delta(t-4T) + \cdots].$$

25.21. (a)

$$\mathcal{X}(z) = \frac{1}{z} + \frac{2}{z^2} - \frac{1}{z^3}.$$

(b)

$$\mathcal{X}(z) = 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots = \frac{1}{1+1/z} = \frac{z}{z+1} \text{ (from (5.4a))}.$$

(c)

$$\begin{aligned}\mathcal{X}(z) &= \{1, \frac{1}{2}, \frac{1}{2^2}, \ldots\} = 1 + \frac{1}{2z} + \frac{1}{2^2z^2} + \cdots \\ &= \frac{1}{1-(1/2z)} = \frac{2z}{2z-1}.\end{aligned}$$

from (5.4a).
(d) Put $z = \mathrm{e}^{Ts}$ as in (25.22):

$$\mathcal{X}(z) = \frac{1}{z}\frac{1}{1-(1/z^2)} = \frac{z}{z^2-1}.$$

25.22. (a) By sampling at times $t = nT$, $n = 0, 1, 2, \ldots$, we obtain $x(t) = \{nT\}$. Then

$$\mathcal{X}(t) = \frac{T}{z} + \frac{2T}{z^2} + \frac{3T}{z^3} + \cdots.$$

This series can be summed by using the process in Example 25.18 to give

$$\mathcal{X}(z) = \frac{Tz}{(z-1)^2}.$$

(b) As in (a) we obtain

$$\mathcal{X}(z) = \frac{\mathrm{e}^{-T}}{z} + \frac{\mathrm{e}^{-2T}}{z^2} + \cdots = \frac{\mathrm{e}^{-T}}{z}\frac{1}{1-(\mathrm{e}^{-T}/z)}$$

by summing the geometric series.
[Note: (c) and (d) have been deleted from the 2003 reprint.)

25.23. (a) $\mathcal{Y} = \mathcal{G}\mathcal{X}$ in general, so

$$\left(1-\frac{1}{z}\right) = \left(1+\frac{1}{z}\right)\mathcal{G}(z)$$

which gives $\mathcal{G}(z)$ in the closed form

$$\mathcal{G}(z) = \left(1-\frac{1}{z}\right)\Big/\left(1+\frac{1}{z}\right).$$

To obtain $g(t)$, first expand $\mathcal{G}(z)$ in powers of $1/z$ for large z. We have (using (5.4a) for geometric series)

$$\mathcal{G}(z) = \left(1-\frac{1}{z}\right)\left(1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right) = 1-\frac{2}{z}+\frac{2}{z^2}-\frac{2}{z^3}+\cdots.$$

Therefore

$$g(t) = \delta(t) - 2\delta(t-T) + 2\delta(t-2T) - \cdots.$$

(b) As in (a)

$$\left(1+\frac{1}{z}\right) = \left(1+\frac{3}{z^3}\right)\mathcal{G}(z),$$

so

$$\mathcal{G}(z) = \left(1+\frac{1}{z}\right)\Big/\left(1+\frac{3}{z^3}\right).$$

(c)

$$\left(1+\frac{1}{z}\right) = \left(1-\frac{1}{z}\right)\mathcal{G}(z),$$

so that

$$\mathcal{G}(z) = \left(1+\frac{1}{z}\right)\Big/\left(1-\frac{1}{z}\right).$$

(d)

$$\mathcal{Y}(z) = 1 - \frac{1}{z^2} + \frac{1}{z^4} - \cdots = \frac{1}{1+(1/z^2)},$$

$$\mathcal{X}(z) = 1 + \frac{1}{z^2} + \frac{1}{z^4} + \cdots = \frac{1}{1-(1/z^2)}.$$

Since $\mathcal{G}(z) = \mathcal{Y}(z)/\mathcal{X}(z)$,

$$\mathcal{G}(z) = \frac{1-(1/z)}{1+(1/z^2} = \frac{z(z-1)}{z^2+1}.$$

(e) As in (d), we find

$$\mathcal{Y}(z) = \frac{1}{1+(1/z^2)}, \quad \mathcal{X}(z) = \frac{1}{1-(1/z^2)},$$

so that

$$\mathcal{G}(z) = \frac{z^2+1}{z^2-1}.$$

25.24. The transform of $y(t)$, defined by the discrete values

$$\{x_0, x_1 \mathrm{e}^{-CT}, x_2 \mathrm{e}^{-2CT}, \ldots\}$$

is $Y(z)$ given by

$$\begin{aligned} Y(z) &= x_0 + \frac{x_1 \mathrm{e}^{-CT}}{z} + \frac{x_2 \mathrm{e}^{-2CT}}{z^2} + \cdots \\ &= x_0 + x_1 \left(\frac{\mathrm{e}^{-CT}}{z}\right) + x_2 \left(\frac{\mathrm{e}^{-CT}}{z}\right)^2 + \cdots \\ &= \mathcal{X}(\mathrm{e}^{-CT}/z), \end{aligned}$$

where $\mathcal{X}(z) = \{x_0, x_1, x_2, \ldots\}$.

25.25.

(a)

$$\mathcal{X}(z) = x_0 + \frac{x_1}{z} + \frac{x_2}{z^2} + \cdots$$

is the transform of $x(t)$. The transform of $(0, x_0, x_1, \ldots)$ is

$$\frac{x_0}{z} + \frac{x_1}{z^2} + \cdots,$$

and this is equal to $\frac{1}{z}\mathcal{X}(z)$.

(b) The rule (a) applies to $\frac{1}{z}\mathcal{X}(z)$ as it did to $\mathcal{X}(z)$: the introduction of a further zero to move the row along causes the transform to be multiplied by a further factor $\frac{1}{z}$, giving $\frac{1}{z^2}\mathcal{X}(z)$. Perform this procedure N times to introduce N zeros; the transform becomes $\frac{1}{z^N}\mathcal{X}(z)$.

25.26. From the definition

$$\{x_0, x_1, x_2, \ldots\} = x_0 + \frac{x_1}{z} + \frac{x_2}{z^2} + \cdots = \mathcal{X}(z).$$

Put

$$\{x_N, x_{N+1}, \ldots\} = x_N + \frac{x_{N+1}}{z} + \cdots = \mathcal{X}^{(N)}(z) \text{ (say)}.$$

Obviously

$$z^N \mathcal{X}(z) = z^N x_0 + z^{N-1} x_1 + \cdots + x_{N-1} + \mathcal{X}^{(N)}(z).$$

Therefore

$$\mathcal{X}^{(N)}(z) = z^N \mathcal{X}(z) - z^N x_0 - z^{N-1} x_1 - \cdots - z x_{N-1}.$$

25.27. *Stability*: the system is stable only if the poles of $\mathcal{G}(z)$ all have modulus less than 1.

(a) $\mathcal{G}(z) = (z+1)/(z^2-4)$. In the argand diagram the poles are at $z = \pm 2 = c_1, c_2$. Since $|c_1|$ (or $|c_2|$) > 1, the system is unstable.

(b) $\mathcal{G}(z) = (z^2 - z)/(4z^2 - 1)$. There are poles at $z = c_1 = \frac{1}{2}$ and $z = c_2 = -\frac{1}{2}$. Their moduli are both < 1, so the system is stable.

(c) $\mathcal{G}(z) = 1/(4z^2 + 1)$. The poles are at $c_1 = \frac{1}{4}\mathrm{i}$ and $c_2 = -\frac{1}{4}\mathrm{i}$. $|c_1|$ and $|c_2|$ are both < 1, so the system is stable.

(d) $\mathcal{G}(z) = (z^3 + 1)/(2z^4 + 5z^2 + 2)$. The poles occur where $2z^4 + 5z^2 + 2 = 0$. Solving this as a quadratic equation for z^2, we obtain $z^2 = -\frac{1}{2}$ and -2, so the poles are at $z = \pm\frac{1}{\sqrt{2}}\mathrm{i}$ and $\pm\sqrt{2}\mathrm{i}$. The poles at $\pm\sqrt{2}\mathrm{i}$ have $|\pm\sqrt{2}\mathrm{i}| = \sqrt{2} > 1$, so the system is unstable.

Growth rate of transients. From the results leading up to eqn (25.40) the overall growth (or decay) of responses of the system is determined by the behaviour of a factor $|c|^N$, where c is the pole associated with a particular mode or transient, and $N = t/T$, where t is time and T the period. For the given functions $\mathcal{G}(z)$, we have :

(a) The factor is 2^N for both poles.

(b) The factor is $(\frac{1}{2})^N$ for both poles.

(c) The factor is $(\frac{1}{4})^N$ for both poles.

(d) The factors are $(\sqrt{2})^N = 2^{\frac{1}{2}N}$ for two poles, and $2^{-\frac{1}{2}N}$ for the other two.

25.28. *Notes.* For the solution method see Example 25.23, or use the general result from Problem 25.26. The notation '→' stands for the words "has the z transform equal to". The counter n runs through $n = 0, 1, 2, \ldots$.

(a) $4y_{n+2} - y_n = x_n$; $y_0 = 1$, $y_1 = 2$. (i)

Put $\{x_n\} = \{x_0, x_1, \ldots\} \to \mathcal{X}(z)$, and $\{y_n\} = \{y_0.y_1, \ldots\} \to \mathcal{Y}(z)$. Then

$$\{y_{n+2}\} \to z^2\mathcal{Y} - z^2 y_0 - z y_1 = z^2\mathcal{Y} - (z^2 + 2z)$$

(by Problem 25.26). The transform of (i) is therefore given by

$$4[z^2\mathcal{Y} - (z^2 + 2z)] - \mathcal{Y} = \mathcal{X},$$

so that

$$\mathcal{Y}(z) = \frac{4(z^2 + 2z) + \mathcal{X}(z)}{4z^2 - 1} = 1 + \frac{1 + 8z + \mathcal{X}(z)}{4z^2 - 1}. \qquad \text{(ii).}$$

The poles occur at $z = \pm\frac{1}{2}$, for which $|z| < 1$. Inspection of (ii) shows that there are no growth or non-diminishing terms arising from an impulsive input, so the system is stable.

(b) $y_{n+2} - 3y_{n+1} + 2y_n = 2xn$; $y_0 = 0$, $y_1 = 1$. (ii)

Proceding as in (a), (ii) becomes

$$(z^2\mathcal{Y} - z) + 3z\mathcal{Y} + 2\mathcal{Y} = 2\mathcal{X},$$

so that

$$\mathcal{Y}(z) = \frac{2\mathcal{X}(z) + z}{(z+1)(z+2)}.$$

Since one of the poles has a modulus > 1, the system is unstable.

(c) $2y_{n+2} + y_{n+1} + y_n = x_{n+1} - x_n$; $y_0 = 0$, $y_1 = 1$. (iii)

Proceeding as in (a), (iii) becomes

$$2(z^2\mathcal{Y} - z) + z\mathcal{Y} + \mathcal{Y} = z\mathcal{X} - zx_0 - \mathcal{X},$$

so that

$$\mathcal{Y}(z) = \frac{(z-1)\mathcal{X}(z) - zx_0 + 2z}{2z^2 + z + 1}.$$

The poles are at $z = \frac{1}{4}(-1 \pm \mathrm{i}\sqrt{7})$, whose moduli are equal to $1/\sqrt{2} < 1$. The argument in Section 25.10 leads us to expect stability; that is, no terms arising from an impulsive input whose effect does not decrease to zero.

(d) $2y_{n+2} + 3y_{n+1} - y_n = x_n$; $y_0 = 1$, $y_1 = 1$. (iv Proceeding as in (a), (iv) becomes

$$(2z^2 + 3z - 1)\mathcal{Y} - (2z^2 + 2z + 3z) = \mathcal{X},$$

or

$$\mathcal{Y}(z) = \frac{\mathcal{X}(z) + 2z^2 + 5z}{2z^2 + 3z - 1}.$$

There are poles at $\frac{1}{4}[-3 \pm \sqrt{17}]$ with modulus > 1, so the system is unstable.

Chapter 26: Fourier series

26.1. These functions are all odd 2π-periodic, which means that $a_0 = a_1 = a_2 = \ldots = 0$, whilst

$$b_n = \frac{2}{\pi}\int_0^\pi f(t)\sin nt\mathrm{d}t.$$

The figures show the graph of the function over the interval $-\pi < t < \pi$.
(a)

$$f(t) = \begin{cases} -1 & (-\pi < t < 0) \\ 1 & (0 \le t \le \pi) \end{cases}$$

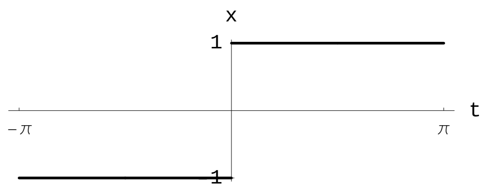

Figure 1: Problem 26.1a

Therefore

$$b_n = \frac{2}{\pi}\int_0^\pi \sin nt\mathrm{d}t = \begin{cases} 4/(n\pi) & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

The Fourier series for $f(t)$ is

$$\frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + \cdots.$$

(b) $f(t) = t$. Then

$$b_n = \frac{2}{\pi}\int_0^\pi t\sin nt\mathrm{d}t = -\frac{2(-1)^n}{n}.$$

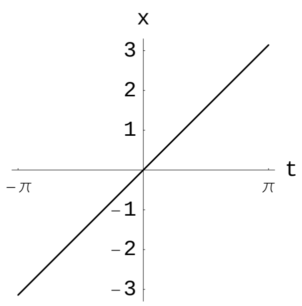

Figure 2: Problem 26.1b

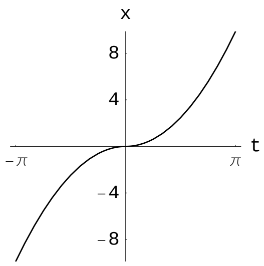

Figure 3: Problem 26.1c

(c)

$$f(t) = \begin{cases} -t^2 & (-\pi < t < 0) \\ t^2 & (0 \le t \le \pi) \end{cases}$$

Therefore

$$b_n = \frac{2}{\pi}\int_0^{\pi} t^2 \sin nt\mathrm{d}t = \frac{2}{n^3\pi}\{2 + (-1)^n[n^2\pi^2 - 2]\}.$$

(d)

$$f(t) = \begin{cases} \mathrm{e}^{-t} - 1 & (-\pi < t < 0) \\ -(\mathrm{e}^t - 1) & (0 \le t \le \pi) \end{cases}$$

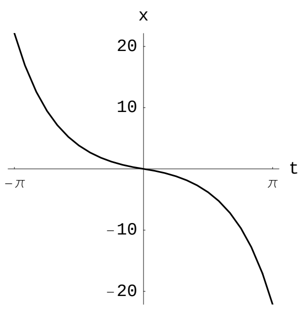

Figure 4: Problem 26.1d

Therefore

$$b_n = -\frac{2}{\pi}\int_0^{\pi} (\mathrm{e}^t - 1)\sin nt\mathrm{d}t = -\frac{2[-1 + (-1)^n + (-1)^n(1 - \mathrm{e}^{\pi})n^2]}{n(1 + n^2)\pi}.$$

(e)

$$f(t) = \begin{cases} 1 & (-\pi < t \le -\frac{1}{2}\pi) \\ -1 & (-\frac{1}{2}\pi < t \le 0 \\ 1 & (0 < t \le \frac{1}{2}\pi) \\ -1 & (\frac{1}{2}\pi < t \le \pi) \end{cases}$$

Therefore

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sin nt\mathrm{d}t - \frac{2}{\pi}\int_{\frac{1}{2}\pi}^{\pi} \sin nt\mathrm{d}t \\ &= \frac{2}{\pi}\left\{\left[-\frac{\cos nt}{n}\right]_0^{\frac{1}{2}\pi} - \left[-\frac{\cos nt}{n}\right]_{\frac{1}{2}\pi}^{\pi}\right\} \\ &= \frac{2}{\pi n}[1 - 2\cos\frac{1}{2}\pi n + (-1)^n]. \end{aligned}$$

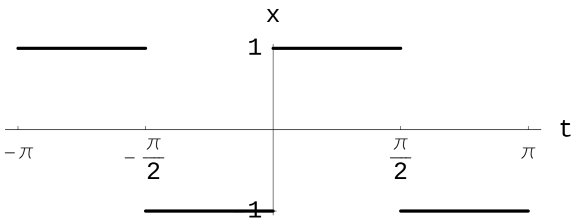

Figure 5: Problem 26.1e

The sequence of coefficients is

$$b_1 = 0, b_2 = \frac{4}{\pi}, b_3 = b_4 = b_5 = 0, b_6 = \frac{4}{3\pi}, b_7 = b_8 = b_9 = 0, b_{10} = \frac{4}{5\pi}, \ldots.$$

26.2. For even functions the coefficients b_n are all zero, whilst for 2π-periodic functions

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\mathrm{d}t, \quad (n = 0, 1, 2, \ldots).$$

The diagrams show the functions over one period $-\pi < t < \pi$.
(a)

$$f(t) = \begin{cases} -1 & (-\pi < t \le \frac{1}{2}\pi) \\ 1 & (-\frac{1}{2}\pi < t \le \frac{1}{2}\pi) \\ -1 & (\frac{1}{2}\pi < t \le \pi) \end{cases}$$

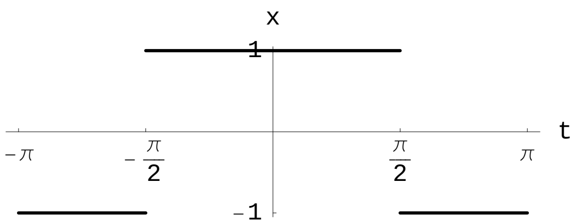

Figure 6: Problem 26.2a

Then

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\mathrm{d}t = \frac{4}{n\pi}\sin\frac{1}{2}n\pi.$$

The even coefficients $a_0, a_2, a_4, \ldots$ are zero, whilst

$$a_1 = \frac{4}{\pi}, \quad a_3 = -\frac{4}{3\pi}, \quad a_5 = \frac{4}{5\pi}, \ldots.$$

(b) $f(t) = t^2$, $(-\pi < t \le \pi)$..
In this case

$$a_n = \frac{2}{\pi}\int_0^{\pi} t^2\cos nt\mathrm{d}t = \frac{4(-1)^n}{n^2}, (n \ge 1)$$

integrating by parts. For $n = 0$, $a_0 = 2\pi^2/3$.
(c) $f(t) = \cos\frac{1}{2}t$, $(-\pi < t \le \pi)$.
Then

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos\frac{1}{2}t\cos nt dt = -\frac{4\cos n\pi}{(4n^2-1)\pi} = -\frac{4(-1)^n}{(4n^2-1)}, \quad (n = 0, 1, 2, \ldots).$$

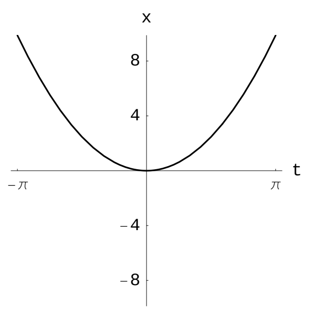

Figure 7: Problem 26.2b

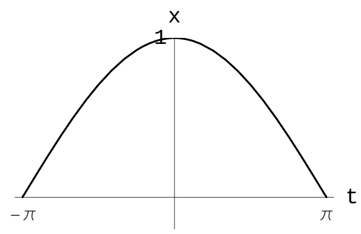

Figure 8: Problem 26.2c

26.3. The Fourier coefficients for a 2π-periodic function $f(t)$ are

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\mathrm{d}t, \quad (n = 0, 1, 2, \ldots).$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\mathrm{d}t. \quad (n = 1, 2, \ldots).$$

(a)

$$f(t) = \begin{cases} 0 & (-\pi < t < 0) \\ t & (0 \le t \le \pi) \end{cases} .$$

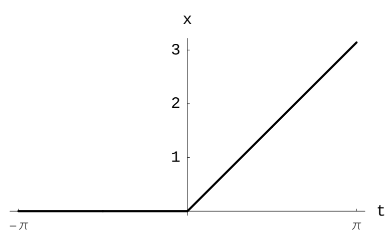

Figure 9: Problem 26.3a

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_0^{\pi} t\mathrm{d}t = \frac{\pi}{,} \quad a_n = \frac{1}{\pi}\int_0^{\pi} t\cos nt\mathrm{d}t = \frac{(-1)^n - 1}{n^2\pi}, \quad (n = 1, 2, \ldots)$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} t\sin nt\mathrm{d}t = -\frac{(-1)^n}{n}, \quad (n = 1, 2, \ldots).$$

(b)

$$f(t) = \begin{cases} t + \pi & (-\pi < t < 0) \\ t & (0 < t \le \pi) \end{cases} .$$

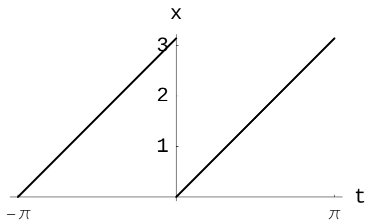

Figure 10: Problem 26.3b

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\mathrm{d}t = \pi, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\mathrm{d}t = 0, \quad (n = 1, 2, \ldots),$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\mathrm{d}t = -\frac{1+(-1)^n}{n}.$$

26.4. The half-rectified sine wave is generated by the 2π-periodic function

$$f(t) = \begin{cases} 0 & (-\pi < t \le 0) \\ \sin t & (0 < t \le \pi) \end{cases}.$$

The Fourier coefficients are

$$a_0 = \frac{2}{\pi}, \quad a_1 = 0, \quad a_n = -\frac{1+(-1)^n}{(n^2-1)\pi} \quad (n = 2, 3, 4, \ldots),$$

$$b_1 = \frac{1}{2}, \quad a_n = 0, \quad (n = 2, 3, 4, \ldots).$$

The Fourier series is

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{(2^2-1)\pi}\cos 2t - \frac{2}{(4^2-1)\pi}\cos 4t - \cdots.$$

26.5. The Fourier coefficients of the 2π-periodic function

$$f(t) = \begin{cases} 0 & (-\pi < t \le 0) \\ 1 & (0 < t \le \pi) \end{cases}$$

are

$$a_0 = 1, \quad a_n = \frac{1}{\pi}\int_0^{\pi} \cos nt\mathrm{d}t = 0 \quad (n = 1, 2, 3, \ldots),$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin nt\mathrm{d}t = \frac{1-(-1)^n}{n} \quad (n = 1, 2, 3, \ldots).$$

Hence $b_1 = 2/\pi$, $b_2 = 0$, $b_3 = 2/(3\pi)$, $b_4 = 0$, $b_5 = 2/(5\pi)$, and so on, which generates the Fourier series

$$\frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right).$$

At $t = 0$, the Fourier series takes the value $\frac{1}{2}$, in agreement with the definition $f(0) = 0$. There is a discontinuity in $f(t)$ at $t = 0$. By (26.12), the Fourier series takes the average value of the left and right hand values of the function at the discontinuity.

Put $t = \frac{1}{2}\pi$. Then $\sin\frac{1}{2}\pi = 1$, $\sin\frac{3\pi}{2} = -1$, $\sin\frac{5\pi}{2} = 1$ and so on. Since the Fourier series equals the function on continuous parts of the curve,

$$f(\tfrac{1}{2}\pi) = 1 = \tfrac{1}{2} + \tfrac{2}{\pi}\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots\right).$$

Therefore

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}.$$

26.6. The fully rectified sine wave of period 2π is $f(t)=F|\sin t|$. This is an even function so that $b_n=0$. The other Fourier coefficients are

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin t|\cos nt\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\pi}\sin t\cos nt\mathrm{d}t=-\frac{2(1+(-1)^n}{(n^2-1)\pi}, \quad (n=0,1,2,\ldots). \end{aligned}$$

Hence

$$|\sin t|=\frac{2}{\pi}-\frac{4}{3\pi}\cos 2t-\frac{4}{15\pi}\cos 4t-\cdots.$$

The amplitude of the first non-zero harmonic (period π) is $4/(3\pi)$.

26.7. The Fourier series is

$$\sum_{n=1}^{\infty}\frac{n+a}{n^3+an+3}\sin nt.$$

The first two harmonics have amplitudes $(a+1)/(a+3)$ and $(a+2)/(2a+11)$. These are in the ratio $2:1$ if

$$\frac{a+1}{a+3}\frac{2a+11}{a+2}=2.$$

Therefore

$$(a+1)(2a+11)=2(a+3)(a+2), \text{ or } 2a^2+13a+11=2(a^2+5a+6).$$

Hence $a=\frac{1}{3}$.

The next harmonic ($n=3$) has amplitude

$$\frac{3+a}{27+3a+3}=\frac{3+\frac{1}{3}}{27+1+3}=\frac{10}{93}.$$

26.8. (See Section 26.9) Note that both functions are odd with period 2π. From Figure 26.18(a) the straight line is given by $x=Ft/\pi$. Its Fourier series is given essentially by Problem 26.1(b), with coefficients (suitably scaled)

$$a_n=0, \quad (n=0,1,2,\ldots), \quad b_n=-\frac{2F(-1)^n}{n\pi}, \quad (n=1,2,\ldots).$$

The second step function can be obtained from Problem 26.1(a) by inserting a minus sign. Its Fourier coefficients are

$$c_n=0, \quad (n=0,1,2,\ldots), \quad d_n=\begin{cases} -4/(n\pi) & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

The Fourier series is therefore

$$\sum_{n=1}^{\infty}(b_n+d_n)\sin nt=\frac{2F-4}{\pi}\sin t-\frac{F}{\pi}\sin 2t+\frac{2F-4}{3\pi}\sin 3t-\frac{F}{2\pi}\sin 4t+\cdots.$$

The leading harmonic has zero amplitude if $F=2$.

26.9. The T-periodic function is $Q(t)=\frac{1}{4}T^2-t^2$ for $-\frac{1}{2}T\le t\le\frac{1}{2}T$, and it is even. Hence all $b_n=0$. The other Fourier coefficients are

$$a_0=\frac{2}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T}(\tfrac{1}{4}T^2-t^2)\mathrm{d}t=\frac{T^2}{6}.$$

$$a_n = \frac{2}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} (\tfrac{1}{4}T^2 - t^2)\cos(2\pi nt/T)\mathrm{d}t = -\frac{(-1)^n T^2}{n^2\pi^2}, \quad (n = 1, 2, 3, \ldots).$$

The approximation given by the first four terms of the Fourier series is

$$\begin{aligned} Q_4(t) &= = \frac{1}{2}a_0 + a_1\cos(2\pi t/T) + a_2\cos(4\pi t/T) + a_3\cos(6\pi t/T) \\ &= T^2\left[\frac{1}{6} + \frac{1}{\pi^2}\cos(2\pi t/T) - \frac{1}{4\pi^2}\cos(4\pi t/T) + \frac{1}{9\pi^2}\cos(6\pi t/T)\right]. \end{aligned}$$

(a) At $t = 0$, $Q(0) = \frac{1}{4}T^2 = 0.25T^2$ and $Q_4(0) = (\frac{1}{6} + \frac{31}{36\pi^2})T^2 = 0.2539\ldots T^2$.
(b) At $t = \frac{1}{4}T$, $Q(\frac{1}{4}T) = \frac{3}{16}T^2 = 0.1875T^2$ and $Q_4(\frac{1}{4}T) = (\frac{1}{6} + \frac{1}{4\pi^2})T^2 = 0.1919\ldots T^2$.

26.10. The 2π-periodic function

$$f(t) = \begin{cases} \beta t(\pi - t) & (0 < t \leq \pi) \\ \beta t(\pi + t) & (-\pi < t \leq 0) \end{cases}.$$

is odd, so that all Fourier coefficients a_n are zero. The other coefficients are

$$b_n = \frac{2}{\pi}\int_0^{\pi} \beta t(\pi - t)\sin nt\mathrm{d}t = \frac{4(1 - (-1)^n)\beta}{n^3\pi},$$

using integration by parts. The Fourier series is

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \frac{4(1 - (-1)^n)\beta}{n^3\pi}\sin nt \\ &= \frac{8\beta}{\pi}\sin t + \frac{8\beta}{27\pi}\sin 3t + \frac{8\beta}{125\pi}\sin 5t + \cdots. \end{aligned}$$

The ratio of the first and third harmonics is

$$\frac{b_1}{b_3} = \frac{8\beta}{\pi}\frac{27\pi}{8\beta} = 27.$$

Let

$$f_3(t) = \frac{8\beta}{\pi}\sin t + \frac{8\beta}{27\pi}\sin 3t.$$

Then comparison of $f(t)$ and $f_3(t)$ at $t = \frac{1}{2}\pi$ gives

$$f(\tfrac{1}{2}\pi) = \tfrac{\pi^2\beta}{4} = 2.467\ldots\beta, \quad f_4(\tfrac{1}{2}\pi) = \tfrac{8\beta}{\pi} - \tfrac{8\beta}{27\pi} = 2.452\ldots\beta.$$

26.11. The function $f(t) = t(\pi^2 - t^2)$ is an odd function which means that all coefficients $a_n = 0$. The sine coefficients are given by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t(\pi^2 - t^2)\sin nt\mathrm{d}t = -\frac{12(-1)^n}{n^3}.$$

The Fourier series of $f(t)$ is

$$f(t) = \sum_{0}^{\infty} b_n \sin nt = -\sum_{n=0}^{\infty} \frac{12(-1)^n}{n^3}\sin nt. \qquad \text{(i)}$$

The derivative of $f(t)$ is $f'(t) = \pi^2 - 3t^2$. As expected $f'(t)$ is an even function so that the coefficients $b_n = 0$. The cosine coefficients are

$$a_0 = 0, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi^2 - 3t^2)\cos nt\mathrm{d}t = -\frac{12(-1)^n}{n^2}.$$

The Fourier series of $f'(t)$ is

$$-\sum_{n=1}^{\infty} \frac{12(-1)^n}{n^2} \cos nt.$$

It can be seen that this Fourier series is the derivative of the Fourier series in (i), which confirms that, in this case, the derivative of the Fourier series of $f(t)$ is the Fourier series of the derivative of $f(t)$.

The function $g(t) = t^3$ is odd so that all the Fourier coefficients a_n are zero. The sine coefficients are given by

$$b_n = -\frac{2(-1)^n}{n^3}(n^2\pi^2 - 6), \quad (n = 1, 2, \ldots)$$

Hence

$$g(t) = \sum_{n=1}^{\infty} \frac{2(-1)^n(6 - n^2\pi^2)}{n^3} \sin nt.$$

The derivative $g'(t) = 3t^2$ which is an even function. Hence all $b_n = 0$, and

$$a_0 = 2\pi^2, \quad a_n = \frac{12(-1)^n}{12n^2}, \quad (n = 1, 2, \ldots)$$

Therefore

$$g'(t) = 3t^2 = 2\pi^2 + \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^2} \cos nt.$$

Clearly the derivative of the Fourier series of $g(t)$ obtained by term-by-term differentiation, namely,

$$\sum_{n=1}^{\infty} \frac{2(-1)^n(6 - n^2\pi^2)}{n^2} \cos nt$$

is not the Fourier series of $g'(t)$. A problem arises because the series $\sum_{n=1}^{\infty} \cos nt$ which occurs among the terms in this expression does not converge. This series does not have a sum.

26.12. A 2π-periodic rectified sine wave is defined by

$$x = P(t) = \begin{cases} 0 & (-\pi \le t \le 0) \\ |\sin 2t| & (0 < t \le \pi) \end{cases}$$

(see Fig. 11).

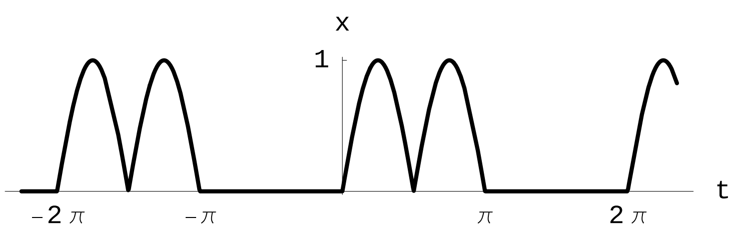

Figure 11: Problem 26.12

The Fourier coefficients are given by

$$a_0 = \frac{4}{\pi}, \quad a_n = \int_0^{\pi} |\sin 2t| \cos nt \mathrm{d}t = -\frac{4}{(n^2-4)\pi}[-(n+1) - 4\cos \tfrac{1}{2}n\pi],$$

$$b_n = \int_0^{\infty} |\sin 2t| \sin nt \mathrm{d}t = -\frac{8}{(n^2-4)} \sin \tfrac{1}{2}n\pi.$$

26.13. One particular representation of the function $f(t) = t$ on $-\pi < t \leq \pi$ is obtained by considering its periodic extension (see Section 26.7). The Fourier coefficients are $a_n = 0, (n = 0, 1, 2, \ldots)$, and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin nt \mathrm{d}t = -\frac{2(-1)^n}{n}.$$

Hence

$$t = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nt,$$

on $-\pi < t \leq \pi$, as required.

Integrate this series term-by-term from $t = 0$ to $t = x$:

$$\begin{aligned}
\int_0^x t \mathrm{d}t &= \frac{1}{2}x^2 = \int_0^x 2 \sum_{n=1}^{x} \frac{(-1)^{n-1}}{n} \sin nt \mathrm{d}t \\
&= -2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} [\cos nt]_{t=0}^{x} \\
&= -2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos nx + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.
\end{aligned}$$

Therefore, for $-\pi < x \leq \pi$,

$$x^2 = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} - 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos nx. \tag{i}$$

Equation (26.10) states that the average value of the function is equal to $\frac{1}{2}a_0$. The average value of x^2 over a period is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \mathrm{d}x = \frac{1}{3}\pi^2.$$

Therefore the constant term in (i) is given by

$$4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{3}\pi^2.$$

This inverse method determines the sum of the series on the left.

26.14. The Fourier series for t^2 can be obtained by referring back to Problem 26.13. Quoting the result

$$t^2 = \frac{1}{3}\pi^2 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt.$$

Integrate both sides of this equation:

$$\begin{aligned}
\int_0^x t^2 \mathrm{d}t = \frac{1}{3}x^3 &= \frac{1}{3}\pi^2 \int_0^x \mathrm{d}t + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_0^x \cos nt \mathrm{d}t \\
&= \frac{1}{3}\pi^2 x + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} [\sin nt]_0^x \\
&= \frac{1}{3}\pi^2 x + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin nx
\end{aligned}$$

Finally

$$x^3 - \pi^2 x = 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin nx.$$

26.15. The T-period function

$$P(t) = \begin{cases} -2t & (-\frac{1}{2}T \leq t < 0) \\ 2t & (0 \leq t < \frac{1}{2}T) \end{cases}$$

is even, so that the Fourier coefficients b_n are all zero. The cosine coefficients are given by

$$a_0 = T, \quad a_n = \frac{4}{T}\int_0^{\frac{1}{2}T} 2t\cos(2\pi nt/T)\mathrm{d}t = \frac{2((-1)^n - 1)T}{n^2\pi^2} \quad (n = 1, 2, \ldots).$$

Hence

$$\begin{aligned} P(t) &= \frac{1}{2}T + \frac{2T}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos(2\pi nt/T) \\ &= \frac{1}{2}T - \frac{4T}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos[2\pi(2m+1)t/T]. \end{aligned}$$

The spectral components are $|\frac{1}{2}a_0|, |a_1|, |a_3|, |a_5| \ldots$, that is

$$\frac{T}{2}, \frac{4T}{\pi^2}, \frac{4T}{9\pi^2}, \frac{4T}{25\pi^2}, \ldots,$$

at $n = 0, 1, 3, 5, \ldots$. The spectrum is shown in Fig. 12.

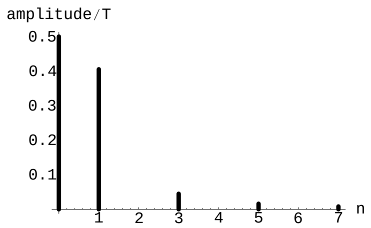

Figure 12: Problem 26.15

26.16. Use formula (26.15) with $f(t) = 1$ and $t_0 = 1$.
(a) For the half-range sine series, the coefficients are

$$b_n = 2\int_0^1 \sin(n\pi t)\mathrm{d}t = \frac{2[1 - (-1)^n]}{n\pi}.$$

Therefore

$$1 = \sum_{n=1}^{\infty}\frac{2[1-(-1)^n]}{n\pi}\sin(n\pi t) = \frac{4}{\pi}\sum_{r=1}^{\infty}\frac{\sin(2r-1)\pi t}{2r-1}.$$

for $0 < t < 1$. Note that this series represents an odd function which takes the value -1 for $-1 < t < 0$, and (by (26.12)) zero at $t = 0$. It is shown in Fig. 13.
(b) For the half-range cosine series, the coefficients are

$$a_0 = 2, \quad a_n = 2\int_0^1 \cos(n\pi t)\mathrm{d}t = \frac{2}{n\pi}[\sin(n\pi t)]_0^1 = 0.$$

This represents an even function which is 1 for all t. In other words the function is its own half-range cosine series

26.17. Use formula (26.15) with $f(t) = t$ and $t_0 = 1$

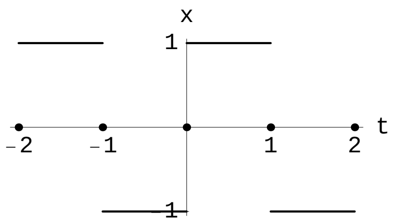

Figure 13: Problem 26.16a

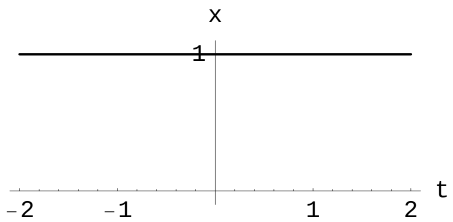

Figure 14: Problem 26.16b

(a) For the half-range cosine series, the coefficients are

$$a_0 = 2\int_0^1 t\mathrm{d}t = 1, \quad a_n = 2\int_0^1 t\cos(n\pi t)\mathrm{d}t = \frac{2((-1)^n - 1)}{n^2\pi^2}.$$

Hence

$$t = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos(n\pi t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{r=1}^{\infty}\frac{\cos[(2r-1)\pi t]}{(2r-1)^2}.$$

(b) For the half-range sine series, the coefficients are

$$b_n = 2\int_0^1 t\sin(n\pi t)\mathrm{d}t = \frac{2(-1)^{n+1}}{n\pi}.$$

Therefore the half-range sine series is

$$t = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(n\pi t),$$

for $0 < t < 1$. For $-1 < t < 0$ the sum of the series is $-t$.

26.18. Use formula (26.15) with $f(t) = \sin\omega t$ and $t_0 = \pi/\omega$. The coefficients for the half-range cosine series are

$$a_0 = \frac{4}{\pi}, \quad a_1 = 0,$$

$$\begin{aligned} a_n &= \frac{2\omega}{\pi}\int_0^{\pi/\omega} f(t)\cos(n\omega t)\mathrm{d}t = \frac{2\omega}{\pi}\int_0^{\pi/\omega}\sin(\omega t)\cos(n\omega t)\mathrm{d}t \\ &= -\frac{2(1 + (-1)^n}{\pi(n^2 - 1)}, \qquad (n = 2, 3, 4, \ldots) \end{aligned}$$

Hence

$$\sin\omega t = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(1 + (-1)^n)}{n^2 - 1}\cos(n\omega t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{r=1}^{\infty}\frac{\cos(2r\omega t)}{4r^2 - 1},$$

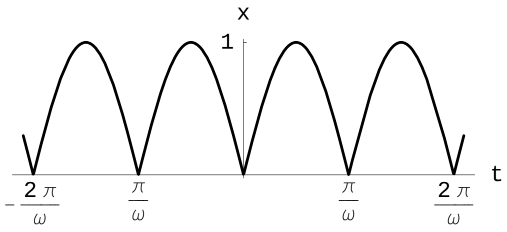

Figure 15: Problem 26.18

for $0 < t < \pi/\omega$. The series represents an even function with sum $|\sin \omega t|$, which is shown in Fig. 15.

26.19. Use formula (26.15) with $f(t) = \cos \omega t$ and $t_0 = \pi/\omega$. The coefficients for the half-range sine series are

$$b_1 = 0, \quad b_n = \frac{2\omega}{\pi} \int_0^{\pi/\omega} \cos(\omega t) \sin(n\omega t) \mathrm{d}t = \frac{2n(1 + (-1)^n)}{\pi(n^2 - 1)}.$$

Hence

$$\cos \omega t = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{n(1 + (-1)^n)}{n^2 - 1} \sin(n\omega t) = \frac{8}{\pi} \sum_{r=1}^{\infty} \frac{r}{4r^2 - 1} \sin(2r\omega t),$$

for $0 < t < \pi/\omega$. The sum of the series is odd, so that for $-\pi/\omega < t < 0$ the sum is $-\cos \omega t$ except at $t = n\pi/\omega$, where its sum is zero.

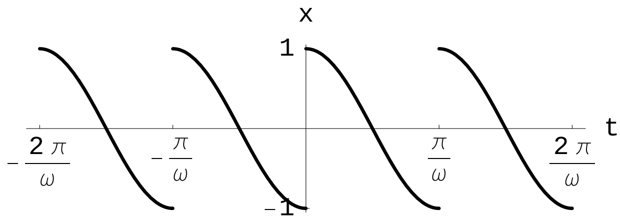

Figure 16: Problem 26.19

26.20. Use formula (26.15) with $f(t) = \cos t$ and $t_0 = 2\pi$. The coefficients for the half-range sine series are

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \cos t \sin(nt/2) \mathrm{d}t = -\frac{2n[(-1)^n - 1]}{\pi(n^2 - 1)},$$

provided $n \neq 2$. For $n = 2$, $b_2 = 0$. Hence

$$\cos t = \frac{4}{\pi} \sum_{r=1}^{\infty} \frac{\sin[\frac{1}{2}(2r - 1)t]}{(2r - 1)^2 - 4},$$

for $0 < t < 2\pi$.

26.21. Use formula (26.15) with $f(t) = \cos t$ and $t_0 = 2\pi$.The coefficients for the half-range cosine series are given by

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} \cos t \mathrm{d}t = 0, \quad a_2 = \frac{1}{\pi} \int_0^{2\pi} \cos^2 t \mathrm{d}t = 1,$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \cos t \cos(nt/2) \mathrm{d}t = 0, \quad (n = 1, 3, 4, \ldots).$$

The function $f(t) = \cos t$ is its own half-range cosine series.

26.22. The function $f(t)$ is given by

$$f(t) = \begin{cases} 1 & (0 \le t < \frac{1}{2}\pi) \\ 0 & (\frac{1}{2}\pi \le t \le \pi) \end{cases}$$

In (26.15), $t_0 = \pi$. (a) For the half-range sine series the coefficients are

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sin nt\mathrm{d}t = \frac{2}{n\pi}[1 - \cos(\tfrac{1}{2}n\pi)]. \end{aligned}$$

Hence the half-range sine series is

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{[1 - \cos\frac{1}{2}n\pi]}{n}\sin nt.$$

The terms for which $n = 4, 8, 12, \ldots$ are all zero.
(b) For the half-range cosine series the coefficients are $a_0 = 1$ and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos nt\mathrm{d}t = \frac{2}{n\pi}\sin\frac{1}{2}n\pi. \end{aligned}$$

Hence

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{1}{2}n\pi}{n}\cos nt = \frac{2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{2r-1}\cos(2r-1)t.$$

26.23. (See Section 26.9) The 2π-periodic function specified by

$$P(t) = \begin{cases} -t & (-\pi \le t \le 0) \\ t & (0 \le t \le \pi) \end{cases}$$

has the Fourier series

$$P(t) = \frac{1}{2}\pi - \frac{4}{\pi}\left(\frac{\cos t}{1^2} + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \cdots\right).$$

(a) Rescale t by putting $t = \pi\tau/2$ so that when $t = \pi$, $\tau = 2$. Hence

$$P(\pi\tau/2) = \frac{1}{2}\pi - \frac{4}{\pi}\left(\frac{\cos(\pi\tau/2)}{1^2} + \frac{\cos(3\pi\tau/2)}{3^2} + \frac{\cos(5\pi\tau/2)}{5^2} + \cdots\right).$$

Finally, rescaling P,

$$\begin{aligned} Q(\tau) &= \frac{6}{\pi}P(\pi\tau/2) = \begin{cases} -3\tau & (-2 \le \tau \le 0) \\ 3\tau & (0 \le t \le 2) \end{cases} \\ &= 3 - \frac{24}{\pi^2}\left(\frac{\cos(\pi\tau/2)}{1^2} + \frac{\cos(3\pi\tau/2)}{3^2} + \frac{\cos(5\pi\tau/2)}{5^2} + \cdots\right). \end{aligned}$$

The symbol τ can be replaced by t in the answer.
(b) First rescale t by putting $t = \pi\tau$, so that when $t = \pi$, $\tau = 1$. Therefore

$$P(\pi\tau) = \frac{1}{2}\pi - \frac{4}{\pi}\left(\frac{\cos\pi\tau}{1^2} + \frac{\cos 3\pi\tau}{3^2} + \frac{\cos 5\pi\tau}{5^2} + \cdots\right).$$

Then

$$R(\tau) = 1 - \frac{1}{\pi}P(\pi\tau) = \frac{1}{2} + \frac{4}{\pi}\left(\frac{\cos\pi\tau}{1^2} + \frac{\cos 3\pi\tau}{3^2} + \frac{\cos 5\pi\tau}{5^2} + \cdots\right).$$

As before τ can be replaced by t.

(c) $P(t)$ has the spectral components

$$\pi/2, 4/\pi, 4/(9\pi), 4/(25\pi), \ldots \text{ at } n = 0, 1, 3, 5, \ldots.$$

The functions $Q(t)$ and $R(t)$ have spectral components at the same values of n but scaled in magnitude.

26.24. (See Section 26.9) The period-2 function

$$P(t) = \begin{cases} -1 & (-1 \le t < 0) \\ 1 & (0 \le t < 1) \end{cases}$$

has the Fourier series

$$P(t) = \frac{2}{\pi}\left(\sin\pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t + \cdots\right).$$

Let $t = 2\tau/T$ so that when $t = 1$, $\tau = \frac{1}{2}T$. Hence

$$P(2\tau/T) = \frac{2}{\pi}\left(\sin(2\pi\tau/T) + \frac{1}{3}\sin(6\pi\tau/T) + \frac{1}{5}\sin(10\pi\tau/T) + \cdots\right).$$

Multiply both sides by a and change the sign. Then

$$\begin{aligned} Q(\tau) &= -aP(2\tau/T) = \begin{cases} a & (\frac{1}{2}T \le t < \frac{1}{2}T) \\ -a & (0 \le 0) \end{cases} \\ &= \frac{2a}{\pi}\left(\sin(2\pi\tau/T) + \frac{1}{3}\sin(6\pi\tau/T) + \frac{1}{5}\sin(10\pi\tau/T) + \cdots\right) \end{aligned}$$

This function is periodic with period $2a$, which means that $Q(\tau) = a$ for $\frac{1}{2}T \le t < T$.

26.25. The 2π-periodic function $f(t) = t$, $(-\pi \le t \le \pi)$ is odd so that the coefficients a_n are all zero. The sine coefficients are given by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\mathrm{d}t = -\frac{2(-1)^n}{n}, \quad (n = 1, 2, 3, \ldots).$$

Hence

$$t = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt.$$

To determine a particular solution of

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \Omega^2 x = K\sin\omega t,$$

try a solution $x = A\cos\omega t$. Then

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \Omega^2 x - K\sin\omega t &= -A\omega^2\cos\omega t + \Omega^2 A\cos\omega t - K\cos\omega t \\ &= [A(\Omega^2 - \omega^2) - K]\cos\omega t = 0 \end{aligned}$$

for all t, if $A = K/(\Omega^2 - \omega^2)$. The forced solution is therefore

$$x = \frac{K\cos\omega t}{\Omega^2 - \omega^2}, \quad (\Omega^2 \ne \omega^2)$$

Consider the general term in the Fourier series for $f(t)$, namely

$$2\frac{(-1)^{n+1}}{n}\sin nt.$$

Comparison with the particular solution just found thus generates a forcing term (with $K = 2(-1)^{n+1}/n$, $\omega = n$),

$$\frac{2(-1)^{n+1}\cos nt}{\Omega^2 - n^2},$$

provided $\Omega^2 \neq n^2$. We now sum these terms over n to give the forced solution

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\Omega^2 - n^2}\cos nt.$$

The system will resonate if Ω is close to any positive integer n.

26.26. Multiply both sides of

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos\omega t + b_n\sin\omega t)$$

by $f(t)$ and integrate term-by-term over the interval $(-\frac{1}{2}T, \frac{1}{2}T)$:

$$\begin{aligned}
&\int_{-\frac{1}{2}T}^{\frac{1}{2}T} f(t)^2\mathrm{d}t \\
&= \frac{1}{2}a_0\int_{-\frac{1}{2}T}^{\frac{1}{2}T} f(t)\mathrm{d}t + \sum_{n=0}^{\infty}\left(a_n\int_{\frac{1}{2}T}^{\frac{1}{2}T} f(t)\cos\omega t\mathrm{d}t + b_n\int_{\frac{1}{2}T}^{\frac{1}{2}T} f(t)\sin\omega t\mathrm{d}t\right) \\
&= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2),
\end{aligned}$$

using (26.9) for the Fourier coefficients.

(a) With $T = \pi$ and the odd function

$$f(t) = \begin{cases} -1 & (-\frac{1}{2}\pi < t \leq 0) \\ 1 & (0 < t \leq \frac{1}{2}\pi) \end{cases},$$

the Fourier coefficients $a_n = 0$ and

$$b_n = \frac{4}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin 2nt\mathrm{d}t = \frac{2(1-(-1)^n)}{n\pi}.$$

Hence

$$b_1 = \frac{4}{\pi},\ b_2 = 0,\ b_3 = \frac{4}{3\pi},\ b_4 = 0,\ b_5 = \frac{4}{5\pi},\ldots.$$

Hence, using Parseval's identity,

$$\begin{aligned}
\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) &= \frac{16}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m+1)^2} \\
&= \frac{2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(t)^2\mathrm{d}t = \frac{2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\mathrm{d}t \\
&= 2.
\end{aligned}$$

Therefore

$$\sum_{m=1}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

(b) The Fourier coefficients of $f(t) = t$, $(-\pi < t \leq \pi)$ are

$$a_n = 0, \quad b_n = \frac{2(-1)^{n+1}}{n}$$

(see Problem 26.1b). Using Parseval's identity

$$\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \mathrm{d}t = \frac{1}{3\pi}[t^3]_{-\pi}^{\pi} = \frac{2\pi^2}{3}.$$

Hence

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

26.27. As far as the Laplace transform is concerned, the transform is that of the function $f(t)\mathrm{H}(t)$, which is zero for $t < 0$. The Laplace transforms of $\cos n\omega t$ and $\sin n\omega t$ are

$$\mathcal{L}\{\cos n\omega t\} = \frac{s}{s^2 + n^2\omega^2}, \qquad \mathcal{L}\{\sin n\omega t\} = \frac{n\omega}{s^2 + n^2\omega^2}.$$

Hence

$$F(s) = \mathcal{L}\{f(t)\} = \frac{a_0}{2s} + \sum_{n=1}^{\infty}\frac{a_n s + b_n n\omega}{s^2 + n^2\omega^2}.$$

The Fourier coefficients of the function

$$f(t) = \begin{cases} -t^2 & (-\pi < t < 0) \\ t^2 & (0 \leq t \leq \pi) \end{cases}$$

are

$$a_n = 0, \qquad b_n = \frac{2}{n^3\pi}\{2 + (-1)^n[n^2\pi^2 - 2]\}.$$

Hence the Fourier transform of the function $f(t)\mathrm{H}(t)$ is

$$F(s) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2 + (-1)^n[n^2\pi^2 - 2]}{n^2(s^2 + n^2)}.$$

26.28. A radio wave is defined by

$$x(t) = a\cos\omega t\cos\omega_0 t,$$

where ω_0 is very much greater than ω. Using the product formula in Appendix B(d),

$$x(t) = \tfrac{1}{2}a[\cos(\omega_0 - \omega)t + \cos(\omega_0 + \omega)t].$$

The figure shows the wave $x(t) = \cos t\cos(10t)$ for $-20 < t < 20$.

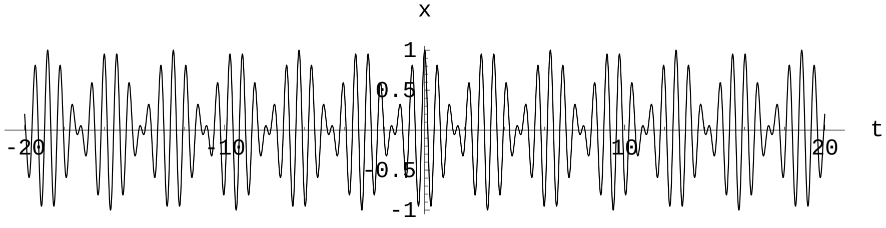

Figure 17: Problem 26.28

(a) With $\omega = 500$ and $\omega_0 = 100001$, the sums and differences are

$$\omega_0 + \omega = 100501, \qquad \omega_0 - \omega = 99501.$$

The greatest common divisor of these numbers is 1: hence the period of $x(t)$ is 2π.
(b) If $\omega = p/q$ and $\omega_0 = r/s$, then

$$x(t) = a\cos(pt/q)\cos(rt/s).$$

Then $x(t)$ is periodic with period T if T is the smallest values for which $x(t+T) = x(t)$ for all t. For the given function

$$x(t+T) = a\cos\left(\frac{p(t+T)}{q}\right)\cos\left(\frac{r(t+T)}{s}\right).$$

This equals $x(t)$ if pT/q and rT/s are integer multiples of 2π. Since q and s have only the common divisor 1 the smallest value of T is $2\pi qs$, which is the period of $x(t)$.

As the sum of two waves

$$x(t) = a\cos\left(\frac{pt}{q}\right)\cos\left(\frac{rt}{s}x\right) = \frac{a}{2}\left[\cos\left(\frac{r}{s} - \frac{p}{q}\right)t + \cos\left(\frac{r}{s} + \frac{p}{q}\right)t\right].$$

This is the Fourier cosine series of $x(t)$ over the period $2\pi sq$.
(c) Given $x_1(t) = \cos t\cos\sqrt{2}t$, then

$$x(t+T) = \cos(t+T)\cos(\sqrt{2}t + \sqrt{2}T).$$

In this case we require T and $\sqrt{2}T$ to be integer multiples of 2π which is impossible since by definition $\sqrt{2}$ can never be equal to the ratio of two integers.

26.29. (a) If $m = n$ then

$$\int_{-\frac{1}{2}T}^{\frac{1}{2}T} e^{i2\pi nf_0t}e^{-i2\pi mf_0t}dt = \int_{-\frac{1}{2}T}^{\frac{1}{2}T} dt = T.$$

If $m \neq n$ and $f_0 = 1/T$, then

$$\begin{aligned}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} e^{i2\pi nf_0t}e^{-i2\pi mf_0t}dt &= \int_{-\frac{1}{2}T}^{\frac{1}{2}T} e^{i2\pi(n-m)f_0t}dt = \left[\frac{Te^{2\pi it(n-m)/T}}{2\pi i(n-m)}\right]_{-\frac{1}{2}T}^{\frac{1}{2}T} \\ &= \frac{T\sin 2\pi(n-m)}{\pi(n-m)} = 0\end{aligned}$$

(b) From (26.18a),

$$x_P(t) = \sum_{n=-\infty}^{\infty} X_n e^{i2\pi nf_0t}.$$

Multiply both sides by $e^{-i2\pi Nf_0t}$ and integrate over one period, from $t = -\frac{1}{2}T$ to $\frac{1}{2}T$:

$$\int_{-\frac{1}{2}T}^{\frac{1}{2}T} x_P(t)e^{i2\pi Nt/T}dt = \sum_{n=-\infty}^{\infty} X_n e^{i2\pi nt/T}e^{-i2\pi Nt/T}dt = X_NT = X_N/f_0.$$

Result (26.18) follows.

26.30. For $x_P(t) = t/T$, the coefficients of the two-sided Fourier series are, from (26.18), given by

$$X_0 = \frac{f_0}{T}\int_0^T t dt = \frac{1}{2}T,$$

$$X_n = \frac{f_0}{T}\int_0^T t\mathrm{e}^{-\mathrm{i}2\pi n f_0 t}\mathrm{d}t = \frac{\mathrm{i}}{2n\pi}.$$

Hence

$$\frac{t}{T} = \frac{1}{2} + \frac{\mathrm{i}}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n}\mathrm{e}^{\mathrm{i}2\pi nt/T}.$$

Chapter 26: Fourier transforms

27.1 The real and imaginary parts of the integral

$$2\int_0^\infty \mathrm{e}^{-t}\mathrm{e}^{2\pi\mathrm{i}ft}\mathrm{d}t$$

give the Fourier cosine and sine transforms $X_c(f)$ and $X_s(f)$ respectively of e^{-t}:

$$\begin{aligned} X_c + \mathrm{i}X_s &= 2\int_0^\infty \mathrm{e}^{t(-1+2\pi\mathrm{i}f)}\mathrm{d}t = \frac{2}{-1+2\pi\mathrm{i}f}[\mathrm{e}^{t(-1+2\pi\mathrm{i}f)}]_0^\infty \\ &= \frac{2}{1-2\pi\mathrm{i}f} = \frac{2(1+2\pi\mathrm{i}f)}{(1-2\pi\mathrm{i}f)(1+2\pi\mathrm{i}f)} = \frac{2(1+2\pi\mathrm{i}f)}{1+(2\pi f)^2} \\ &= \frac{2+4\pi\mathrm{i}f}{1+4\pi^2f^2} \end{aligned}$$

Therefore

$$X_c = \frac{2}{1+4\pi^2f^2}, \quad X_s = \frac{4\pi f}{1+4\pi^2f^2}.$$

The inverse is

$$I(t) = 2\int_0^\infty \frac{4\pi f}{1+4\pi^2f^2}\sin(2\pi ft)\mathrm{d}f.$$

$I(t)$ is zero at $t=0$. This is connected with (27.9). $I(t)$ is an odd function on $-\infty < t < \infty$; $I(t) = \mathrm{e}^{-t}$ for $t > 0$ and $I(t) = -\mathrm{e}^{-t}$ for $t < 0$, so it jumps in value at $t_0 = 0$, the mean of the values at either side of $t = 0$ being zero.

27.2. The cosine transform of

$$x(t) = \begin{cases} 1-t, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

is

$$X_c(f) = 2\int_0^\infty x(t)\cos(2\pi ft)\mathrm{d}t = 2\int_0^1 (1-t)\cos(2\pi ft)\mathrm{d}t.$$

By integrating by parts we obtain

$$X_c = \frac{2}{(2\pi f)^2}[1-\cos(2\pi f)] = \frac{4\sin^2(\pi f)}{(2\pi f)^2} = \frac{\sin^2(\pi f)}{\pi^2 f^2} = \mathrm{sinc}^2 f.$$

The inverse of $X_c(f)$ is

$$2\int_0^\infty \frac{\sin^2(\pi f)}{\pi^2f^2}\cos(2\pi ft)\mathrm{d}f.$$

This is an even function of t; it delivers $(1-t)$ for $t > 0$, and $(1+t)$ for $t < 0$, and is continuous at $t = 0$, taking the value 1. Therefore put $t = 0$ into the inverse integral; we obtain

$$2\int_0^\infty \frac{\sin^2(\pi f)}{\pi^2f^2}\mathrm{d}f = 1.$$

Now put $\pi f = u$; then the equality becomes

$$2\int_0^\infty \frac{\sin^2 u}{u^2}\frac{\mathrm{d}u}{\pi} = 1, \text{ or } \int_0^\infty \frac{\sin^2 u}{u^2}\mathrm{d}u = \frac{1}{2}\pi.$$

27.3. Here, $x(t) = \mathrm{e}^{-t^2}$, and we require the cosine transform $X_c(f)$, where

$$X_c(f) = 2\int_0^\infty \mathrm{e}^{-t^2}\cos(2\pi f t)\mathrm{d}t.$$

By differentiating with respect to f under the integral sign

$$\frac{\mathrm{d}X_c}{\mathrm{d}f} = -4\pi\int_0^\infty (t\mathrm{e}^{-t^2})\sin(2\pi f t)\mathrm{d}t. \qquad \text{(i)}$$

Integrate by parts, putting into eqn (17.8) $u = \sin(2\pi f t)$, and $\mathrm{d}v/\mathrm{d}t = t\mathrm{e}^{-t^2}$ so that $v = -\frac{1}{2}\mathrm{e}^{-t^2}$:

$$\begin{aligned}\frac{\mathrm{d}X_c}{\mathrm{d}f} &= -4\pi\left\{\left[\sin(2\pi f t)\left(-\tfrac{1}{2}\mathrm{e}^{-t^2}\right)\right]_0^\infty - \int_0^\infty(-\tfrac{1}{2}\mathrm{e}^{-t^2})[2\pi f\cos(2\pi f t)]\mathrm{d}t\right\}\\ &= -4\pi^2 f\int_0^\infty \mathrm{e}^{-t^2}\cos(2\pi f t)\mathrm{d}t = -2\pi^2 f X_c.\end{aligned}$$

The differential equation

$$\frac{\mathrm{d}X_c}{\mathrm{d}f} = -2\pi^2 f X_c \qquad \text{(ii)}$$

is separable:

$$\int\frac{\mathrm{d}X_c}{X_c} = -2\pi^2\int f\mathrm{d}f,$$

leading to the general solution

$$X_c(f) = K\mathrm{e}^{-\pi^2 f^2}$$

where K is an arbitrary constant. The inverse integral is therefore

$$x(t) = 2K\int_0^\infty \mathrm{e}^{-\pi^2 f^2}\cos(2\pi f t)\mathrm{d}f \qquad \text{(iii)}$$

for some value of K.

To find K, put $t = 0$ into (iii). We have $x(t) = \mathrm{e}^{-t^2}$, so

$$x(0) = 1 = 2K\int_0^\infty \mathrm{e}^{-\pi^2 f^2}\mathrm{d}f = \frac{2K}{\pi}\int_0^\infty \mathrm{e}^{-u^2}\mathrm{d}u = \frac{2K}{\pi}\frac{\sqrt{\pi}}{2} = \frac{K}{\sqrt{\pi}},$$

from the standard integral given. Therefore $K = \sqrt{\pi}$, and

$$\mathrm{e}^{-t^2} \leftrightarrow \sqrt{\pi}\mathrm{e}^{-\pi^2 f^2}.$$

27.4. (a) $x(t)$ is an even function, that is $x(t) = x(-t)$. The Fourier transform is given by

$$\mathcal{F}[x(t)] = \int_{-\infty}^\infty x(t)\mathrm{e}^{-2\pi\mathrm{i}ft}\mathrm{d}t = \int_0^\infty x(t)\mathrm{e}^{-2\pi\mathrm{i}ft}\mathrm{d}t + \int_{-\infty}^0 x(t)\mathrm{e}^{-2\pi\mathrm{i}ft}\mathrm{d}t.$$

Change the variable in the second integral by putting $t = -t'$, and put $x(-t') = x(t')$ (evenness property):

$$\begin{aligned}\mathcal{F}[x(t)] &= \int_0^\infty x(t)\mathrm{e}^{-2\pi\mathrm{i}ft}\mathrm{d}t + \int_0^\infty x(t')\mathrm{e}^{2\pi\mathrm{i}ft'}\mathrm{d}t'\\ &= 2\int_0^\infty x(t)\frac{\mathrm{e}^{2\pi\mathrm{i}ft} + \mathrm{e}^{-2\pi\mathrm{i}ft}}{2}\mathrm{d}t\\ &= 2\int_0^\infty x(t)\cos(2\pi f t).\mathrm{d}t \qquad \text{(i)}\end{aligned}$$

The right-hand side of (i) is the Fourier cosine transform $X_c(f)$.
(b) The function e^{-t^2} is even, so eqn (i) applies. We then have

$$\mathcal{F}[\mathrm{e}^{-t^2}] \equiv X_c(f) = \sqrt{\pi}\mathrm{e}^{-\pi^2 f^2}.$$

By the time-scaling rule (27.18b),

$$\mathcal{F}[\mathrm{e}^{-\alpha t^2}] \equiv \mathcal{F}[\mathrm{e}^{-(\alpha^{\frac{1}{2}} t)^2}] = \frac{1}{\alpha^{\frac{1}{2}}} X_c\left(\frac{f}{\alpha^{\frac{1}{2}}}\right) = \sqrt{\frac{\pi}{\alpha}}\mathrm{e}^{-\pi^2 f^2 \alpha}.$$

(Notice that if $\alpha = \pi$, the result becomes symmetrical: $\mathrm{e}^{-\pi t^2} \leftrightarrow \mathrm{e}^{-\pi f^2}$.

27.5. With $x(t)$ an odd, real function

$$X(f) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-2\pi \mathrm{i} f t}\mathrm{d}t = \int_{0}^{\infty} x(t)\mathrm{e}^{-2\pi \mathrm{i} f t}\mathrm{d}t + \int_{-\infty}^{0} x(t)\mathrm{e}^{-2\pi \mathrm{i} f t}\mathrm{d}t.$$

Change the variable in the second integral to $t' = -t$, and use the oddness property, $x(-t') = -x(t')$:

$$\begin{aligned} X(f) &= \int_0^\infty x(t)\mathrm{e}^{-2\pi \mathrm{i} f t}\mathrm{d}t - \int_0^\infty x(t')\mathrm{e}^{2\pi \mathrm{i} f t'}\mathrm{d}t' \\ &= \int_0^\infty x(t)\{\mathrm{e}^{-2\pi \mathrm{i} f t} - \mathrm{e}^{2\pi \mathrm{i} f t}\}\mathrm{d}t \\ &= -2\mathrm{i}\int_0^\infty x(t)\sin(2\pi f t)\mathrm{d}t = -\mathrm{i}X_s(f) \end{aligned}$$

where $X_s(f)$ is the sine transform of $x(t)$.

27.6. See the answer to Problem (27.4b), in the special case of $\alpha = \pi$.

27.7. To derive a form of Fourier transform pair that is an alternative to that in eqns (27.8).
Change the frequency variable f in eqns (27.8) to a new variable ω through the relation $f = \omega/(2\pi)$ (ω then takes the meaning of circular, or angular frequency). Then (27.8a) becomes

$$x(t) = \int_{-\infty}^{\infty} X\left(\frac{\omega}{2\pi}\right)\mathrm{e}^{\mathrm{i}\omega t}\frac{d\omega}{2\pi}, \tag{i}$$

and (27.8b) becomes

$$X\left(\frac{\omega}{2\pi}\right) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t. \tag{ii}$$

Now put

$$\frac{1}{\sqrt{(2\pi)}} X\left(\frac{\omega}{2\pi}\right) = X_1(\omega).$$

Then (i) and (ii) become

$$x(t) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} X_1(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega,$$

and

$$X_1(\omega) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t,$$

as required

27.8. (This is an alternative approach from that in Problem 27.4.) In (27.8b), change the variable from t to $(-t)$ and use the evenness property of $x(t)$. We then have two alternative forms of the equation:

$$X(f) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-2\pi \mathrm{i} f t}\mathrm{d}t \text{ and } X(f) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{2\pi \mathrm{i} f t}\mathrm{d}t.$$

Add the two versions and divide by 2:

$$\begin{aligned} X(f) &= \frac{1}{2}\int_{\infty}^{\infty} x(t)(\mathrm{e}^{2\pi \mathrm{i} ft} + \mathrm{e}^{-2\pi \mathrm{i} ft})\mathrm{d}t = \int_{-\infty}^{\infty} x(t)\cos(2\pi ft)\mathrm{d}t \\ &= 2\int_{0}^{\infty} x(t)\cos(2\pi ft)\mathrm{d}t, \qquad \text{(i)} \end{aligned}$$

since the integrand is even in t. Also $X(f)$ is even (in f), so to obtain the inverse, start with (27.8a), and carry out a similar process of changing the variable from f to $(-f)$, obtaining

$$x(t) = 2\int_{0}^{\infty} X(f)\cos(2\pi ft)\mathrm{d}f. \qquad \text{(ii)}$$

Equations (i) and (ii) take a real form (though $x(t)$ may be complex), recognizable as a Fourier cosine transform and its inverse.

27.9. The function $x(t)$ is odd (that is, $x(-t) = -x(t)$). In (27.8b) change the variable from t to $(-t)$ and use the oddness property. We now have two alternative forms for $X(f)$:

$$X(f) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-2\pi \mathrm{i} ft}\mathrm{d}t \text{ and } X(f) = -\int_{-\infty}^{\infty} x(t)\mathrm{e}^{2\pi \mathrm{i} ft}\mathrm{d}t.$$

Add the two versions and divide by 2; we obtain

$$\begin{aligned} X(f) &= \frac{1}{2}\int_{-\infty}^{\infty} x(t)(\mathrm{e}^{-2\pi \mathrm{i} ft} - \mathrm{e}^{2\pi \mathrm{i} ft})\mathrm{d}t = -\mathrm{i}\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)\mathrm{d}t \\ &= -2\mathrm{i}\int_{0}^{\infty} x(t)\sin(2\pi ft)\mathrm{d}t, \qquad \text{(i)} \end{aligned}$$

since the integrand is an even function of t.

Notice that $X(f)$ is an odd function of f (that is, $X(-f) = -X(f)$), and consider the inverse integral (27.8a), changing the variable (as before) from f to $(-f)$ so as to obtain a second representation of $x(t)$:

$$x(t) = \int_{-\infty}^{\infty} X(f)\mathrm{e}^{2\pi \mathrm{i} ft}\mathrm{d}f \text{ and } x(t) = -\int_{-\infty}^{\infty} X(f)\mathrm{e}^{-2\pi \mathrm{i} ft}\mathrm{d}f.$$

Add the two versions and divide by 2:

$$\begin{aligned} x(t) &= \frac{1}{2}\int_{-\infty}^{\infty} X(f)(\mathrm{e}^{2\pi \mathrm{i} ft} - \mathrm{e}^{-2\pi \mathrm{i} ft})\mathrm{d}t = \mathrm{i}\int_{-\infty}^{\infty} X(f)\sin(2\pi ft)\mathrm{d}f \\ &= 2\mathrm{i}\int_{0}^{\infty} X(f)\sin(2\pi ft)\mathrm{d}t \qquad \text{(ii)} \end{aligned}$$

Now define a function $X_1(f)$ by writing $\mathrm{i}X(f) = X_1(f)$. Then (i) and (ii) become the pair

$$X_1(f) = 2\int_{0}^{\infty} x(t)\sin(2\pi ft)\mathrm{d}t,$$

and

$$x(t) = 2\int_{0}^{\infty} X_1(f)\sin(2\pi ft)\mathrm{d}f.$$

This is the sine transform pair (eqn (27.5)), and is a real form (though x and X may be complex).

27.10. *Time-scaling rule (27.18b).* Let $x(t) \leftrightarrow X(f)$, and consider the transform of $x(At)$:

$$\mathcal{F}[x(At)] = \int_{-\infty}^{\infty} x(At)\mathrm{e}^{-2\pi \mathrm{i} ft}\mathrm{d}t.$$

Change the variable of integration by putting $At = t'$. If $A > 0$, we obtain

$$\frac{1}{A}\int_{-\infty}^{\infty} x(t')\mathrm{e}^{-2\pi \mathrm{i} f t'/A}\mathrm{d}t' = \frac{1}{A}X\left(\frac{f}{A}\right).$$

If $A < 0$, we obtain (note the inversion of the limits in the integral):

$$\frac{1}{A}\int_{\infty}^{-\infty} x(t')\mathrm{e}^{-2\pi \mathrm{i} f t'/A}\mathrm{d}t' = -\frac{1}{A}X\left(\frac{f}{A}\right).$$

Both results can be combined into one by writing

$$\mathcal{F}[x(At)] = \frac{1}{|A|}X\left(\frac{f}{A}\right).$$

Time-delay rule (27.18c).

$$\mathcal{F}[x(t-B)] = \int_{-\infty}^{\infty} x(t-B)\mathrm{e}^{-2\pi \mathrm{i} f t}\mathrm{d}t.$$

Put $t - B = t'$. The integral becomes

$$\int_{-\infty}^{\infty} x(t')\mathrm{e}^{-2\pi \mathrm{i} f(t'+B)}\mathrm{d}t' = \mathrm{e}^{-2\pi \mathrm{i} f B}X(f).$$

27.11. (a) By the time-delay rule (27.18c), with $B = \frac{1}{2}$,

$$\mathcal{F}[\Pi(t-\tfrac{1}{2})] = \mathrm{e}^{-2\pi \mathrm{i}(1/2)f}\operatorname{sinc} f = \mathrm{e}^{-\pi \mathrm{i} f}\operatorname{sinc} f.$$

(b) To confirm this directly: $\Pi(t-\frac{1}{2})$ is zero when $t - \frac{1}{2} < -\frac{1}{2}$ and $t - \frac{1}{2} > \frac{1}{2}$, so

$$\begin{aligned}\mathcal{F}[\Pi(t-\tfrac{1}{2})] &= \int_0^1 (1)\mathrm{e}^{-2\pi \mathrm{i} f t}\mathrm{d}t = -\frac{1}{2\pi f}[\mathrm{e}^{-2\pi \mathrm{i} f t}]_0^1 \\ &= -\frac{\mathrm{e}^{-2\pi \mathrm{i} f} - 1}{2\pi \mathrm{i} f} = \frac{(\mathrm{e}^{\pi \mathrm{i} f} - \mathrm{e}^{-\pi \mathrm{i} f})\mathrm{e}^{-\pi \mathrm{i} f}}{2\pi \mathrm{i} f} = \mathrm{e}^{-\pi \mathrm{i} f}\frac{\sin \pi f}{\pi f} \\ &= \mathrm{e}^{-\pi \mathrm{i} f}\operatorname{sinc} f.\end{aligned}$$

(c) The graph of $x(t)$ is shown in the figure. It is obtained by moving a centrally-placed rectangle of width c given by $\Pi(t/c)$ a distance $b > \frac{1}{2}c$ to the left, and changing its sign to produce A, and a distance b to the right to produce B.

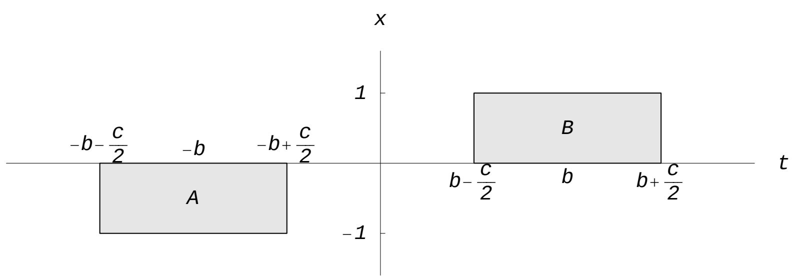

Figure 18: Problem 27.11

Therefore

$$x(t) = -\Pi\left(\frac{t+b}{c}\right) + \Pi\left(\frac{t-b}{c}\right).$$

By the time-delay rule (28.18c), with $B = \pm b$,

$$\begin{aligned}\mathcal{F}[x(t)] &= (-\mathrm{e}^{2\pi \mathrm{i} bf} + \mathrm{e}^{-2\pi \mathrm{i} bf})\mathcal{F}\left[\Pi\left(\frac{t}{c}\right)\right] \\ &= (-\mathrm{e}^{2\pi \mathrm{i} bf} + \mathrm{e}^{-2\pi \mathrm{i} bf})[c\,\mathrm{sinc}\,(cf)] \\ &\qquad \text{(by the time-scaling rule (28.18b) with } B = 1/c) \\ &= -2\mathrm{i}c\sin(2\pi bf)\mathrm{sinc}\,(cf).\end{aligned}$$

27.12. (a) Given that $\Lambda(t) \leftrightarrow \mathrm{sinc}^2 f$, we obtain from the time-scaling rule (27.18b) with $A = 2$:

$$\Lambda(2t) \leftrightarrow \tfrac{1}{2}\mathrm{sinc}^2(\tfrac{1}{2}f).$$

(b) In general, suppose that $x(t) \leftrightarrow X(f)$. To obtain $\mathcal{F}[x(2t-3)]$ we can use the time-scaling rule (27.18b), then the time-delay rule (27.18c), as follows:

$$x(2t) \leftrightarrow \tfrac{1}{2}X(\tfrac{1}{2}f) \quad \text{(time-scaling).}$$

and

$$x(2t-3) = x(2[t-\tfrac{3}{2}]) \leftrightarrow \mathrm{e}^{-2\pi\mathrm{i}(3/2)f}\{\tfrac{1}{2}X(\tfrac{1}{2}f)\} = \tfrac{1}{2}\mathrm{e}^{-3\pi\mathrm{i}f}X(\tfrac{1}{2}f).$$

For the particular case of $x(t) = \Lambda(t)$, $X(f) = \mathrm{sinc}^2 f$, so that we have

$$\Lambda(2t-3) \leftrightarrow \tfrac{1}{2}\mathrm{e}^{-3\pi\mathrm{i}f}\mathrm{sinc}^2(\tfrac{1}{2}f).$$

27.13. (a) *Frequency shift rule (27.18e).* From the definition of $\mathcal{F}$

$$\begin{aligned}\mathcal{F}[x(t)\mathrm{e}^{2\pi\mathrm{i}Dt}] &= \int_{-\infty}^{\infty} x(t)\mathrm{e}^{2\pi\mathrm{i}Dt}\mathrm{e}^{-2\pi\mathrm{i}ft}\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-2\pi\mathrm{i}(f-D)t}\mathrm{d}t = X(f-D)\end{aligned}$$

where $X(f) = \mathcal{F}[x(t)]$.

(b)

$$\mathcal{F}[x(t)\mathrm{e}^{\pm 2\pi\mathrm{i}f_0 t}] = X(f \mp f_0), \text{ (from (a))}$$

(c) From (b)

$$\begin{aligned}\mathcal{F}[x(t)\cos(2\pi f_0 t)] &= \frac{1}{2}\mathcal{F}[x(t)\{\mathrm{e}^{2\pi\mathrm{i}f_0 t} + \mathrm{e}^{-2\pi\mathrm{i}f_0 t}\}] \\ &= \frac{1}{2}\{X((f-f_0) + X(f+f_0)\},.\end{aligned}$$

$$\begin{aligned}\mathcal{F}[x(t)\sin(2\pi f_0 t)] &= \frac{1}{2\mathrm{i}}\mathcal{F}[x(t)\{\mathrm{e}^{2\pi\mathrm{i}f_0 t} - \mathrm{e}^{-2\pi\mathrm{i}f_0 t}\}] \\ &= \frac{1}{2\mathrm{i}}\left\{X((f-f_0) - X(f+f_0)\right\}.\end{aligned}$$

(d) $\mathcal{F}[\Pi(t)] = \mathrm{sinc}\, f$. By the time-scaling rule (27.18b) with $A = \frac{1}{2}$,

$$\mathcal{F}[\Pi(\tfrac{1}{2}t)] = 2\mathrm{sinc}\, 2f.$$

Apply the results in (c); then

$$\mathcal{F}[\Pi(\tfrac{1}{2}t)\cos(2\pi f_0 t)] = \mathrm{sinc}\, 2(f-f_0) + \mathrm{sinc}\, 2(f+f_0),$$

and

$$\mathcal{F}[\Pi(\tfrac{1}{2}t)\sin 2\pi f_0 t)] = -\mathrm{i}\{\mathrm{sinc}\, 2(f-f_0) - \mathrm{sinc}\, 2(f+f_0)\}.$$

27.14. (a) To obtain $\mathcal{F}[\text{sinc}^2 t]$, given that $\mathcal{F}[\Lambda(t)] = \text{sinc}^2 f$. In the duality rule (27.18b) put $X(t) = \text{sinc}^2 t$ and $x(-f) = \Lambda(-f) = \Lambda(f)$ since Λ is an even function. We obtain $\mathcal{F}[\text{sinc}^2 t] = \Lambda(f)$.

This result may alternatively be obtained without using (27.18b). From the basic transform pair, starting with the inverse relation (27.8a), we know that

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} f t} \text{sinc}^2 f \mathrm{d}f = \Lambda(t).$$

This equation is an identity, so if we interchange the letters f and t in it, the result remains true:

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} f t} \mathrm{d}t = \Lambda(f)$$

for all values of the parameter f. Therefore

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i}(-f)t} \text{sinc}^2 t \mathrm{d}t = \Lambda(-f),$$

or

$$\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} f t} \text{sinc}^2 \mathrm{d}t = \Lambda(-f) \equiv \Lambda(f)$$

since Λ is even. Therefore $\mathcal{F}[\text{sinc}^2 t] = \Lambda(f)$.

(b) Given that $\mathcal{F}[\text{sinc}^2 t] = \Lambda(f)$, the time-scaling rule (27.18b) with $A = a$ gives

$$\mathcal{F}[\text{sinc}^2 (at)] = \frac{1}{a} \Lambda \left(\frac{f}{a} \right).$$

Now write $\text{sinc}^2 (at + b) \equiv \text{sinc}^2 [a(t + (b/a))]$, and use the time-delay rule (27.18c) with $B = -b/a$:

$$\mathcal{F}[\text{sinc}^2 (at + b)] = \frac{1}{a} \mathrm{e}^{2\pi \mathrm{i} b f/a} \Lambda \left(\frac{f}{a} \right).$$

27.15. (a) To prove the differentiation rule (27.18h). From (27.8a),

$$x(t) = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} f t} X(f) \mathrm{d}f.$$

By differentiating with respect to t under the integral sign:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} f t} \{2\pi \mathrm{i} f X(f)\} \mathrm{d}f,$$

that is

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \leftrightarrow (2\pi \mathrm{i} f) X(f).$$

As each subsequent differentiation introduces another factor $2\pi \mathrm{i} f$, we have

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} \leftrightarrow (2\pi \mathrm{i} f)^n X(f).$$

(This process can only be carried out so long as the functions on the right continue to have valid Fourier integrals.)

(b) Given $\mathrm{e}^{-|t|} \leftrightarrow 2/(1 + 4\pi^2 f^2)$, to find the inverse of $\mathrm{i} f/(1 + 4\pi^2 f^2)$. The differentiation rule in (a), with $x(t) = \mathrm{e}^{-|t|}$, gives

$$\frac{\mathrm{d}}{\mathrm{d}t} (\mathrm{e}^{-|t|}) =\leftrightarrow \frac{4\pi \mathrm{i} f}{1 + 4\pi^2 f^2}. \qquad \text{(i)}$$

Also

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-|t|}) = \begin{cases} \mathrm{e}^{t} & \text{for } t < 0 \\ -\mathrm{e}^{-t} & \text{for } t > 0 \end{cases},$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-|t|}) = -\mathrm{e}^{-|t|}\mathrm{sgn}\,(t),$$

where sgn (t) stands for the sign of t (see Section 1.4). At $t = 0$ the derivative does not exist. Therefore, from (i), the inverse of $\mathrm{i}f/(1+4\pi^2 f^2)$ is

$$-\frac{1}{4\pi}\mathrm{e}^{-|t|}\mathrm{sgn}\,(t).$$

27.16. Given (eqn (27.17)) that $1/(\alpha + 2\pi\mathrm{i}f)$ is the transform of the function $h(t)$ defined by

$$h(t) = \begin{cases} 0, & t < 0 \\ \mathrm{e}^{-\alpha t}, & t > 0 \end{cases}$$

to obtain $\mathcal{F}[\mathrm{e}^{-\alpha|t|}]$. It is easy to show that

$$\mathrm{e}^{-\alpha|t|} = h(t) + h(-t),$$

by sketching diagrams of $f(t)$ and $f(-t)$. From the time-reversal rule (27.18b),

$$\begin{aligned} \mathcal{F}[\mathrm{e}^{-\alpha|t|}] &= \mathcal{F}[h(t)] + \mathcal{F}[h(-t)] \\ &= \frac{1}{\alpha + 2\pi f} + \frac{1}{\alpha + 2\pi(-f)} = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}. \end{aligned}$$

27.17. (a) $\mathrm{H}(t)$ is the Heaviside unit function, eqn (1.13).

$$\mathcal{F}[\mathrm{e}^{-\alpha t}H(t)] = X(f) = \frac{1}{\alpha + 2\pi\mathrm{i}f},$$

from eqn (27.17). From the modulation rules (27.18f), with $K = \beta/(2\pi)$,

$$\begin{aligned} \mathcal{F}[\mathrm{e}^{-\alpha t}\cos(\beta t)\mathrm{H}(t)] &= \frac{1}{2}\left\{X\left(f - \frac{\beta}{2\pi}\right) + X\left(f + \frac{\beta}{2\pi}\right)\right\} \\ &= \frac{1}{2}\left\{\frac{1}{\alpha + 2\pi\mathrm{i}(f - \frac{\beta}{2\pi})} + \frac{1}{\alpha + 2\pi\mathrm{i}(f + \frac{\beta}{2\pi})}\right\} \qquad \text{(i)} \end{aligned}$$

and

$$\mathcal{F}[\mathrm{e}^{-\alpha t}\sin(\beta t)\mathrm{H}(t)] = \frac{1}{2\mathrm{i}}\left\{\frac{1}{\alpha + 2\pi\mathrm{i}(f - \frac{\beta}{2\pi})} - \frac{1}{\alpha + 2\pi\mathrm{i}(f + \frac{\beta}{2\pi})}\right\} \qquad \text{(ii)}$$

(b) In the case of $\mathcal{F}[\mathrm{e}^{-\alpha t}\cos(2\pi f_0 t + \phi)\mathrm{H}(t)]$, expand the cosine term, obtaining

$$\begin{aligned} &\mathcal{F}[\mathrm{e}^{-\alpha t}\cos(2\pi f_0 t + \phi)\mathrm{H}(t)] = \\ &\qquad \cos\phi\,\mathcal{F}[\mathrm{e}^{-\alpha t}\cos(2\pi f_0 t)\mathrm{H}(t)] - \sin\phi\,\mathcal{F}[\mathrm{e}^{-\alpha t}\sin(2\pi f_0 t)\mathrm{H}(t)]. \end{aligned}$$

The results (i) and (ii) above, with $\beta = 2\pi f_0$, apply to the two terms, giving

$$\begin{aligned} &\frac{1}{2}\cos\phi\left\{\frac{1}{\alpha + 2\pi\mathrm{i}(f - f_0)} + \frac{1}{\alpha + 2\pi\mathrm{i}(f + f_0)}\right\} - \\ &\qquad \frac{1}{2\mathrm{i}}\sin\phi\left\{\frac{1}{\alpha + 2\pi\mathrm{i}(f - f_0)} - \frac{1}{\alpha + 2\pi\mathrm{i}(f + f_0)}\right\} \\ &= \frac{1}{2}\left\{\frac{\mathrm{e}^{\mathrm{i}\phi}}{\alpha + 2\pi\mathrm{i}(f - f_0)} + \frac{\mathrm{e}^{-\mathrm{i}\phi}}{\alpha + 2\pi\mathrm{i}(f + f_0)}\right\} \\ &= \frac{(\alpha\cos\phi - 2\pi f_0\sin\phi) + 2\pi\mathrm{i}f\cos\phi}{\alpha^2 - 4\pi^2(f^2 - f_0^2) + 4\pi\mathrm{i}\alpha f}. \end{aligned}$$

27.18. By (27.27), the notation $x_1(t) * x_2(t)$ means

$$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\mathrm{d}u.$$

Put $x_2 = \mathrm{H}(t)$ and $x_1(t) = x(t)\mathrm{H}(t)$. Then $x_2(\tau)$ is zero for $\tau < 0$ and $x(t-\tau)$ is zero for $t-\tau < 0$, or $\tau > t$. Therefore

$$x1(t) * x_2(t) = \int_0^t x(\tau)\mathrm{d}\tau.$$

27.19. (a)

$$\mathrm{e}^{-t}\mathrm{H}(t) * \mathrm{e}^{-t}\mathrm{H}(t) = \int_{\infty}^{\infty} \mathrm{e}^{-u}\mathrm{e}^{-(t-u)}\mathrm{H}(u)\mathrm{d}u = I(t), \text{ say.}$$

If $t > 0$, $\mathrm{H}(u)\mathrm{H}(t-u)$ is zero for $u < 0$ and $u > t$. If $t < 0$, then $\mathrm{H}(u)\mathrm{H}(t-u)$ is zero for all u. Therefore

$$I(t) = \begin{cases} \int_0^t \mathrm{e}^{-u}\mathrm{e}^{-(t-u)}\mathrm{d}u = \mathrm{e}^{-t}\int_0^t \mathrm{d}u = t\mathrm{e}^{-t} & \text{for } t > 0 \\ \text{zero} & \text{for } t < 0. \end{cases}$$

(b) From (27.17), $\mathcal{F}[\mathrm{e}^{-t}\mathrm{H}(t)] = 1/(1+2\pi\mathrm{i}f)$. From (a)

$$\mathcal{F}[t\mathrm{e}^{-t}\mathrm{H}(t)] = \mathcal{F}[\mathrm{e}^{-t}\mathrm{H}(t) * \mathrm{e}^{-t}\mathrm{H}(t)].$$

Therefore, by the convolution theorem (27.28),

$$\mathcal{F}[t\mathrm{e}^{-t}\mathrm{H}(t)] = \mathcal{F}[\mathrm{e}^{-t}\mathrm{H}(t)]\mathcal{F}[\mathrm{e}^{-t}\mathrm{H}(t)] = \frac{1}{1+2\pi\mathrm{i}f)^2}. \tag{i}$$

(c) $\mathrm{H}(\alpha t) = \mathrm{H}(t)$ for $\alpha > 0$, so the time-scaling rule (27.18b) applied to (i) gives

$$\mathcal{F}[\alpha t\mathrm{e}^{-\alpha t}\mathrm{H}(t)] = \frac{1}{\alpha}\frac{1}{(1+2\pi\mathrm{i}f/\alpha)^2} = \frac{\alpha}{(\alpha+2\pi\mathrm{i}f)^2}.$$

Therefore, dividing through by α,

$$\mathcal{F}[t\mathrm{e}^{-\alpha t}\mathrm{H}(t)] = \frac{1}{(\alpha+2\pi\mathrm{i}f)^2}. \tag{ii}$$

(d) From (27.17), $\mathcal{F}[\mathrm{e}^{-\alpha t}\mathrm{H}(t)] = 1/(\alpha+2\pi\mathrm{i}f)$ for $\alpha > 0$

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}ft}\frac{\mathrm{d}f}{\alpha+2\pi\mathrm{i}f} = \mathrm{e}^{-\alpha t}\mathrm{H}(t).$$

Differentiate both sides with respect to α; we obtain

$$-\int_{-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}f}\frac{\mathrm{d}f}{(\alpha+2\pi\mathrm{i}f)^2} = -t\mathrm{e}^{-\alpha t}\mathrm{H}(t).$$

Therefore

$$t\mathrm{e}^{-\alpha t}\mathrm{H}(t) \leftrightarrow \frac{1}{(\alpha+2\pi\mathrm{i}f)^2},$$

as already found in (ii)

27.20. By (27.15), $\mathcal{F}[\Pi(t)] = \operatorname{sinc} f$. From the time-delay rule (27.18c):

$$\mathcal{F}[t \pm \tfrac{1}{2}] = \mathrm{e}^{\pm\pi\mathrm{i}f}\operatorname{sinc} f.$$

Therefore by the convolution theorem

$$\begin{aligned} \mathcal{F}[\Pi(t-\tfrac{1}{2}) * \Pi(t+\tfrac{1}{2})] &= \mathcal{F}[\Pi(t-\tfrac{1}{2})]\mathcal{F}[\Pi(t+\tfrac{1}{2})] \\ &= (\mathrm{e}^{-\mathrm{i}\pi f}\operatorname{sinc} y)(\mathrm{e}^{\mathrm{i}\pi f}\operatorname{sinc} y) = \operatorname{sinc}^2 f \end{aligned}$$

(b) By proceeding as in (a), we obtain

$$\begin{aligned}\mathcal{F}[\Pi(t-a)] * \Pi(t-b)] &= (\mathrm{e}^{-2\pi \mathrm{i} a f}\mathrm{sinc}\, f)(\mathrm{e}^{-2\pi \mathrm{i} b f}\mathrm{sinc}\, f) \\ &= \mathrm{e}^{-2\pi \mathrm{i}(a+b) f}\mathrm{sinc}^2 f\end{aligned}$$

(c) In the definition (27.27a) put $x_1(t) = \Pi(t)$, $x_2(t) = \Pi(\frac{1}{2}t)$, and use the second form in (27.27a) for simplicity. Then put

$$\Pi(t) * \Pi(\tfrac{1}{2}t) = \int_{-\infty}^{\infty} \Pi(t-u)\Pi(\tfrac{1}{2}u)\mathrm{d}u = (t). \qquad \text{(i)}$$

$\Pi(t-u)$ is zero for $t-u < -\frac{1}{2}$ and for $t-u > \frac{1}{2}$, and $\Pi(\frac{1}{2}u)$ is zerofor $u < -1$ and for $u > 1$. By sketching a diagram as in Example 27.10 we can establish the effective limits of the integral in (i) in terms of t.

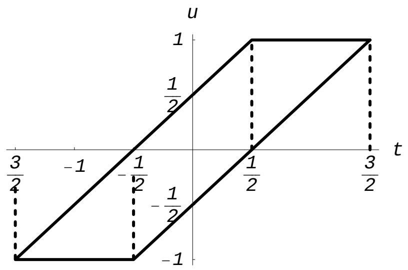

Figure 19: Problem 27.20

Also $F(t)$ is an even function, so we only have to calculate for $t \geq 0$ and the values for $t < 0$ will echo those values. The diagram has boundaries $u = \pm 1$ and $u = t \pm \frac{1}{2}$. The values of $F(t)$ are given in the following table.

Range of t	$F(t)$
$t < -\frac{3}{2}$	zero
$-\frac{3}{2} \leq t \leq -\frac{1}{2}$	$\int_{-1}^{t+\frac{1}{2}} \mathrm{d}u = t + \frac{3}{2}$
$-\frac{1}{2} \leq t \leq \frac{1}{2}$	$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \mathrm{d}u = 1$
$\frac{1}{2} < t \leq \frac{3}{2}$	$\int_{t-\frac{1}{2}}^{1} \mathrm{d}u = 1 - (t - \frac{1}{2}) = -t + \frac{3}{2}$
$t > \frac{1}{2}$	zero

27.21. By Section 27.9, the energy E is given by

$$\begin{aligned}E &= \int_{-\infty}^{\infty} \{\mathrm{e}^{-\alpha t}\mathrm{H}(t)\}^2 \mathrm{d}t = \int_{-\infty}^{\infty} \mathrm{e}^{-2\alpha t}\mathrm{H}(t)\mathrm{d}t, \text{ (since } [\mathrm{H}(t)]^2 = \mathrm{H}(t)) \\ &= \int_{0}^{\infty} \mathrm{e}^{-2\alpha t}\mathrm{d}t = \frac{1}{2\alpha}.\end{aligned}$$

The transform (frequency distribution) of $\mathrm{e}^{-\alpha t}$ is $1/(\alpha + 2\pi f) = X(f)$, say, by (27.17). The energy associated with the frequency range $-f_0 \leq f \leq f_0$ in the expression (27.33) is

$$\begin{aligned}\int_{-f_0}^{f_0} |X(f)|^2 \mathrm{d}f &= \int_{-f_0}^{f_0} \frac{\mathrm{d}f}{|\alpha + 2\pi \mathrm{i} f|^2} = \int_{f_0}^{f_0} \frac{\mathrm{d}f}{\alpha^2 + 4\pi^2 f^2} \\ &= \frac{\alpha}{2\pi}\frac{1}{\alpha^2}\int_{-2\pi f_0/\alpha}^{2\pi f_0/\alpha} \frac{\mathrm{d}u}{1+u^2} \text{ (after putting } f = \alpha u/(2\pi)) \\ &= \frac{1}{2\pi\alpha}[\arctan u]_{-2\pi f_0/\alpha}^{2\pi f_0/\alpha} = \frac{1}{\pi\alpha}\arctan\left(\frac{2\pi f_0}{\alpha}\right).\end{aligned}$$

27.22. The problem asks you to show that if $x(t)$ is zero for $t < -\frac{1}{2}T$ and $t > \frac{1}{2}T$, then $\text{Ш}_T(t)*x(t)$ is periodic with period T. By the convolution theorem

$$\mathcal{F}[\text{Ш}_T(t) * x(t)] = \mathcal{F}[\text{Ш}_T(t)]\mathcal{F}[x(t)]. \tag{i}$$

Here

$$\mathcal{F}[x(t)] = \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \mathrm{e}^{-2\pi \mathrm{i} f t} x(t)\mathrm{d}t,$$

and from (27.32), since $f_0 = 1/T$,

$$\mathcal{F}[\text{Ш}_T(t)] = \frac{1}{T}\,\text{Ш}_{\frac{1}{T}}(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta(f - (n/T)).$$

Therefore (i) becomes

$$\mathcal{F}[\text{Ш}_T(t)] = \sum_{n=-\infty}^{\infty} \left(\frac{1}{T}\int_{-\frac{1}{2}T}^{\frac{1}{2}T} \mathrm{e}^{-2\pi \mathrm{i} f t} x(t)\mathrm{d}t\right) \delta(f - (n/T))$$

which, by (27.21), is the Fourier transform of the T-periodic extension of $x(t)$ to all values of t (remember that $1/T = f_0$ in (27.21)).

27.23. Take the Fourier transform of the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - x = \frac{1}{1+t^2},$$

using the differentiation rule (27.18h) to convert the derivative:

$$\mathcal{F}\left[\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - x\right] = (2\pi \mathrm{i} f)^2 X(f) - X(f) = \mathcal{F}\left[\frac{1}{1+t^2}\right].$$

Therefore

$$X(f) = -\frac{1}{1+4\pi^2 f^2}\mathcal{F}\left[\frac{1}{1+t^2}\right] = -\frac{1}{2}\mathcal{F}[\mathrm{e}^{-|t|}]\mathcal{F}\left[\frac{1}{1+t^2}\right]$$

(by Example 27.3). This is the product of two transforms, so by the convolution theorem the inverse $x(t)$ of $X(f)$ can be written as the convolution integral

$$x(t) = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-|t-u|}}{1+u^2}\mathrm{d}u.$$

(There are, of course, two linearly independent solutions but only one has a Fourier transform.)

27.24. $\Pi(t)$ is continuous at $t = 0$ taking the value 1. Therefore we may put $t = 0$ into the defining Fourier integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} f t} \operatorname{sinc} t \,\mathrm{d}t = \Lambda(t).$$

We obtain

$$\int_{-\infty}^{\infty} \operatorname{sinc} t \mathrm{d}t = 1 = 2\int_0^{\infty} \operatorname{sinc} t \,\mathrm{d}t,$$

since $\operatorname{sinc} t$ is even. Therefore $\int_0^\infty \operatorname{sinc} t \,\mathrm{d}t = \frac{1}{2}$.

(b) $$\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} f t}\mathrm{e}^{-\pi t^2}\mathrm{d}t = \mathrm{e}^{-\pi f^2}, \text{ (see Problem 27.6)}$$

Differentiate both sides of the equation with respect to f:

$$-2\pi\mathrm{i}\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}ft} t\mathrm{e}^{-\pi t^2}\mathrm{d}t = -2\pi f\mathrm{e}^{-\pi f^2},$$

so $\mathcal{F}[t\mathrm{e}^{-\pi t^2}] = -\mathrm{i}f\mathrm{e}^{-\pi f^2}$.

27.25. By the convolution theorem and the result $\mathcal{F}[\operatorname{sinc} t] = \Pi(f)$,

$$\mathcal{F}[\operatorname{sinc} t * \operatorname{sinc} t] = \mathcal{F}[\operatorname{sinc} t]\mathcal{F}[\operatorname{sinc} t] = \Pi^2(f) = \Pi(f).$$

Similarly, and using the time-delay rule (27.18c),

$$\mathcal{F}[\operatorname{sinc}(t-a) * \operatorname{sinc}(t+a)] = (\mathrm{e}^{-2\pi\mathrm{i}af}\Pi(f))(\mathrm{e}^{2\pi\mathrm{i}af}\Pi(f)) = \Pi^2(f) = \Pi(f).$$

27.26. (a) Moving average of $g(t)$. To show that

$$g_\tau(t) = \frac{1}{\tau}\int_{t-\frac{1}{2}\tau}^{t+\frac{1}{2}\tau} g(u)\mathrm{d}u$$

can be written as

$$\frac{1}{\tau}\Pi\left(\frac{t}{\tau}\right) * g(t).$$

The convolution is defined by (27.27):

$$\frac{1}{\tau}\Pi\left(\frac{t}{\tau}\right) * g(t) = \frac{1}{\tau}\int_{-\infty}^{\infty} \Pi\left(\frac{t-u}{\tau}\right) g(u)\mathrm{d}u. \qquad \text{(i)}$$

$\Pi[(t-u)/\tau]$ is zero when $[(t-u)/\tau] < -\frac{1}{2}$ or $[(t-u)/\tau] > \frac{1}{2}$; that is, when $u > t+\frac{1}{2}\tau$ or $u < t-\frac{1}{2}\tau$. Therefore the integral is zero for these ranges, and equals $g(u)$ elsewhere, and (i) becomes

$$\frac{1}{\tau}\int_{t-\frac{1}{2}\tau}^{t+\frac{1}{2}\tau} g(u)\mathrm{d}u = g_\tau(t).$$

(b) The moving average of $\Pi(t)$ over intervals of length τ is given by $F_\tau(t)$, where

$$F_\tau(t) = \frac{1}{\tau}\int_{-\infty}^{\infty} \Pi(t-u)\Pi\left(\frac{u}{\tau}\right)\mathrm{d}u, \qquad \text{(i)}$$

(using (a) with the convolution form (27.26b), with $\Pi(t)$ for $x_1(t)$ and $x_2(t) = \Pi(t/\tau)$, the averaging function). Although this looks more complicated than the direct definition of $g_\tau(t)$, it permits evaluation on the lines of Example 27.10 and Fig. 27.14, which allows extensions to other functions also.

The boundary of the region where the integrand is nonzero (it takes the value 1) is shown by the parallelogram in the figure. The formulae giving the effective limits of integration in (i) change as t passes over the points A', B', B, A. The figure shows a case where $\tau > 1$, together with the values of t at these critical points. When $\tau < 1$ the construction is the same and the formula is the same, but B will be on the negative side of the u axis and B' on the positive side. Both possibilities are covered by expressing the results in the following way:

$$-\tfrac{1}{2}(\tau+1) \le t \le -\tfrac{1}{2}|\tau-1| \quad F_\tau = \tfrac{1}{\tau}\int_{-\frac{1}{2}\tau}^{t+\frac{1}{2}}\mathrm{d}u = \tfrac{1}{\tau}\{(t+\tfrac{1}{2}) - (-\tfrac{1}{2}\tau)\} = \tfrac{1}{\tau}(t+\tfrac{1}{2}\tau+\tfrac{1}{2});$$

$$-\tfrac{1}{2}|\tau-1| \le t \le \tfrac{1}{2}|\tau-1| \quad F_\tau(t) = \tfrac{1}{\tau}\int_{-\frac{1}{2}\tau}^{\frac{1}{2}\tau}\mathrm{d}u = 1;$$

$$\tfrac{1}{2}|\tau-1| \le t \le \tfrac{1}{2}(\tau+1) \quad F_\tau(t) = \tfrac{1}{\tau}\int_{t-\frac{1}{2}}^{\frac{1}{2}\tau}\mathrm{d}u = \tfrac{1}{\tau}(-t+\tfrac{1}{2}\tau+\tfrac{1}{2});$$

(note that, predictably, this is an even function of t.) $F_\tau(t) = 0$ elsewhere.

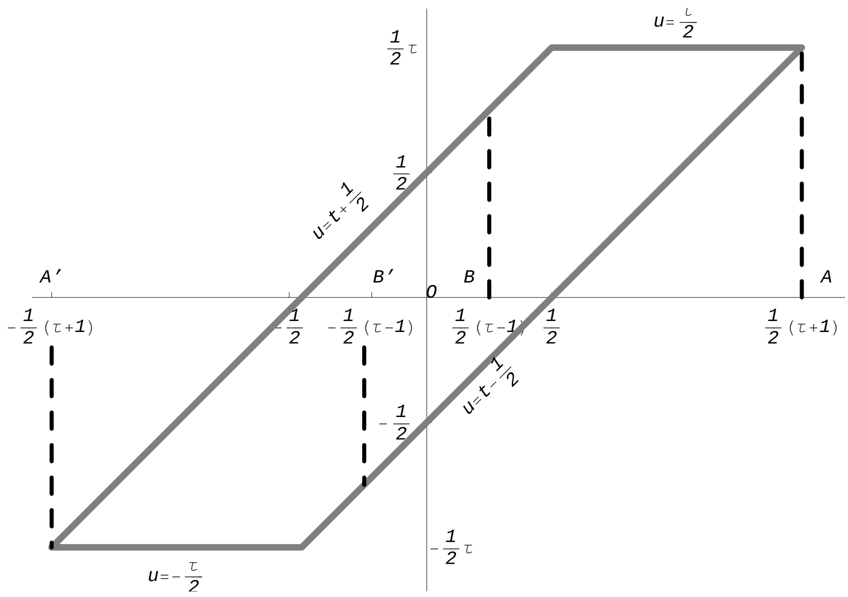

Figure 20: Problem 27.26

27.27. The definition of the convolution is

$$F(t) = \int_{-\infty}^{\infty} g(w+b)\Pi\left(\frac{w+b}{\tau}\right) Ш_T(t-w-a)\mathrm{d}w$$

with $\tau \le T$. Put $w - a = u$:

$$\begin{aligned} F(t) &= \int_{-\infty}^{\infty} g(u+a-b)\Pi\left(\frac{u+a-b}{\tau}\right) Ш_T(t-u)\mathrm{d}u \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ g(u+a-b)\Pi\left(\frac{u+a-b}{\tau}\right)\right\} \delta(t-u-nT)\mathrm{d}u \quad \text{(i)} \end{aligned}$$

This resembles the form in Example 27.11, with the function $x(t)$ displaced along the u axis by a distance $a-b$ to the left, as in the figure.

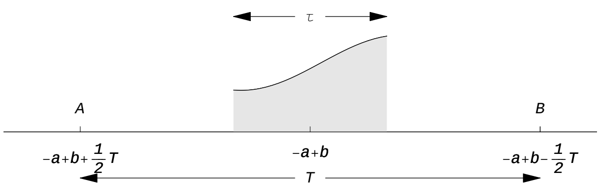

Figure 21: Problem 27.27

The condition $\tau \le T$ ensures that the nonzero segment of the function does not extend beyond the range AB: $-a+b-\frac{1}{2}T \le u \le -a+b+\frac{1}{2}T$. The argument in Example 27.11 can be adapted to fit this case. For a given t, there is always exactly one value of n, say N, such that a single impulse in (i) lies in the interval AB of length T. But if n is any integer

$$(t-nT) + (N+n)T \equiv t - NT,$$

so (from the sifting property) the value picked out is reproduced at $t - nT$, for all n. Therefore the convolution represents the periodic extension, period T, of $g(u + a - b)$ as depicted.

27.28. The function

$$h(t) = \Pi(t) * \Lambda(t) = \int_{-\infty}^{\infty} \Pi(u)\Lambda(t-u)\mathrm{d}u$$

from (27.27). $\Pi(u)$ is zero for $u < \frac{1}{2}$ and $u > \frac{1}{2}$, and $\Lambda(t-u)$ is zero for $t - u < -1$ and $t - u > 1$. To obtain the effective limits of integration, follow the procedure leading to Fig. 27.14 in the book. The boundaries of the region of the (t, u) plane on which the integrand is nonzero are $u = \pm\frac{1}{2}$ and $t - u = \pm 1$, shown in Fig. 22. (The definition of $\Lambda(u)$ is given in Example 27.10.) We have, for

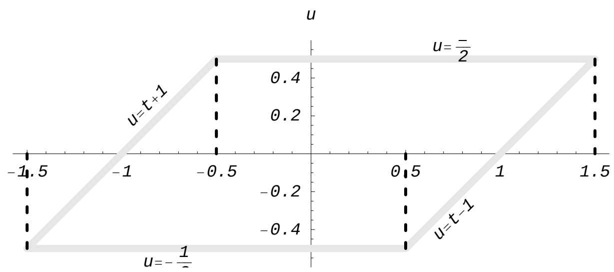

Figure 22: Problem 27.28

$$\begin{array}{rl}
t \le \frac{3}{2} & h(t) = 0; \\
-\frac{3}{2} \le t \le -\frac{1}{2} & h(t) = \int_{-\frac{1}{2}}^{t+1}(1+u)\mathrm{d}u = \frac{1}{2}t^2 + 2t + \frac{15}{8}; \\
-\frac{1}{2} \le t \le 0 & h(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}}(1+u)\mathrm{d}u = 1; \\
0 \le t \le \frac{1}{2} & h(t) = \int_{-\frac{1}{2}}^{\frac{1}{2}}(1-u)\mathrm{d}u = 1; \\
\frac{1}{2} \le t \le \frac{3}{2} & h(t) = \int_{t-1}^{\frac{1}{2}}(1-u)\mathrm{d}u = \frac{1}{2}t^2 + 2t + \frac{15}{8}; \\
t \ge \frac{3}{2} & h(t) = 0.
\end{array}$$

By the convolution Theorem (27.28),

$$\mathcal{F}[h(t)] = \mathcal{F}[\Pi(t)]\mathcal{F}[\Lambda(t)] = \operatorname{sinc} f \operatorname{sinc}^2 f = \operatorname{sinc}^3 f$$

(by Example 27.10). Therefore, since $h(t)$ is continuous (no jumps) the inverse of $\mathcal{F}[h(t)]$ for all t is given by

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} f t} \operatorname{sinc}^3 f \mathrm{d}f = h(t).$$

Put $t = 0$:

$$\int_{-\infty}^{\infty} \operatorname{sinc}^3 f \mathrm{d}f = h(0) = 1.$$

27.29. (a)

$$\begin{aligned}
x(t) * \{Ay(t) + Bz(t)\} &= \int_{-\infty}^{\infty} x(u)\{Ay(t-u) + Bz(t-u)\}\mathrm{d}u \\
&= A\int_{-\infty}^{\infty} x(u)y(t-u)\mathrm{d}u + B\int_{-\infty}^{\infty} x(u)z(t-u)\mathrm{d}u \\
&= Ax(t) * y(t) + Bx(t) * z(t).
\end{aligned}$$

(b) Quoted in (27.26) in the text: here we check it directly:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(u)y(t-u)\mathrm{d}u.$$

Change the variable to $v = t - u$:

$$\begin{aligned} x(t) * y(t) &= -\int_{\infty}^{-\infty} x(t-v)y(v)\mathrm{d}v \\ &= \int_{-\infty}^{\infty} y(v)x(t-v)\mathrm{d}v = y(t) * x(t) \end{aligned}$$

(c) We shall consider the Fourier integrals of the terms, so as to avoid double integrals (Chapter 32). By the convolution theorem (27.28), used twice,

$$\begin{aligned} \mathcal{F}[x(t) * \{y(t) * z(t)\}] &= \mathcal{F}[x(t)]\mathcal{F}[y(t) * z(t)] \\ &= \mathcal{F}[x(t)]\mathcal{F}[y(t)]\mathcal{F}[z(t)]. \end{aligned}$$

The same product appears if the group is bracketed differently (or placed in any order).