The Chemistry Maths Book

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Solutions

Chapter 12 Second-order differential equations.
Constant coefficients

Section 12.2

1. Show that e^{-2x} and $e^{2x/3}$ are particular solutions of the differential equation $3y'' + 4y' - 4y = 0$.

We have $y = e^{-2x} \rightarrow y' = \dfrac{dy}{dx} = -2e^{-2x} = -2y \rightarrow y'' = \dfrac{d^2y}{dx^2} = 4e^{-2x} = 4y$

Therefore $3y'' + 4y' - 4y = \left[12 - 8 - 4\right]y = 0$

Similarly $y = e^{2x/3} \rightarrow y' = \dfrac{2}{3}e^{2x/3} = \dfrac{2}{3}y \rightarrow y'' = \dfrac{4}{9}e^{2x/3} = \dfrac{4}{9}y$

and $3y'' + 4y' - 4y = \left[\dfrac{4}{3} + \dfrac{8}{3} - 4\right]y = 0$

2. Show that e^{3x} and xe^{3x} are particular solutions of the differential equation $y'' - 6y' + 9y = 0$.

$$y = e^{3x} \rightarrow y' = \frac{dy}{dx} = 3e^{3x} = 3y \rightarrow y'' = \frac{d^2y}{dx^2} = 9e^{3x} = 9y$$

Therefore $y'' - 6y' + 9y = \left[9 - 18 + 9\right]y = 0$

$$y = xe^{3x} \rightarrow y' = e^{3x} + 3xe^{3x} \rightarrow y'' = 6e^{3x} + 9xe^{3x}$$

Therefore $y'' - 6y' + 9y = 0 = \left[6 + 9x - 6 - 18x + 9x\right]e^{3x} = 0$

3. Show that $\cos 2x$ and $\sin 2x$ are particular solutions of the differential equation $y'' + 4y = 0$.

$$y = \cos 2x \rightarrow y' = -2\sin 2x \rightarrow y'' = -4\cos 2x = -4y$$

$$y'' + 4y = 0 = \left[-4 + 4\right]y = 0$$

and $y = \sin 2x \rightarrow y' = 2\cos 2x \rightarrow y'' = -4\sin 2x = -4y$

$$y'' + 4y = 0 = \left[-4 + 4\right]y = 0$$

Write down the general solution of the differential equation in

4. Exercise 1: $y = ae^{-2x} + be^{2x/3}$

5. Exercise 2: $y = ae^{3x} + bxe^{3x} = (a+bx)e^{3x}$

6. Exercise 3: $y = a\cos 2x + b\sin 2x$

Section 12.3

Find the general solutions of the differential equations:

7. $y'' - y' - 6y = 0$

The characteristic equation of the differential equation is

$$\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$$
$$= 0 \text{ when } \lambda = 3 \text{ and } \lambda = -2$$

Two particular solutions of the differential equation are therefore

$$y_1 = e^{3x}, \qquad y_2 = e^{-2x}$$

and, because these functions are linearly independent, the general solution is

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^{-2x}$$

8. $2y'' - 8y' + 3y = 0$

The characteristic equation is

$$2\lambda^2 - 8\lambda + 3 = 0 \text{ when } \lambda = \frac{8 \pm \sqrt{64-24}}{4} = 2 \pm 2\sqrt{5}$$

The general solution of the differential equation is therefore

$$y = c_1 e^{(2+2\sqrt{5})x} + c_2 e^{(2-2\sqrt{5})x} = e^{2x}\left[c_1 e^{2\sqrt{5}x} + c_2 e^{-2\sqrt{5}x}\right]$$

9. $y'' - 8y' + 16y = 0$

The characteristic equation

$$\lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0$$

has the double root $\lambda = 4$. Two particular solutions are therefore e^{4x} and xe^{4x}, and the general solution is

$$y(x) = (c_1 + c_2 x)e^{4x}$$

10. $4y'' + 12y + 9y = 0$

The characteristic equation

$$4\lambda^2 + 12\lambda + 9 = (2\lambda + 3)^2$$
$$= 0 \text{ when } x = -3/2 \text{ (double root)}$$

The general solution of the differential equation is therefore

$$y(x) = (c_1 + c_2 x)e^{-3x/2}$$

11. $y'' + 4y' + 5y = 0$

The characteristic equation is

$$\lambda^2 + 4\lambda + 5 = 0$$

with roots $\lambda = \frac{1}{2}\left(-4 \pm \sqrt{16 - 20}\right) = -2 \pm i$

The two particular solutions,

$$y_1(x) = e^{(-2+i)x} \quad \text{and} \quad y_2(x) = e^{(-2-i)x}$$

are linearly independent, and the general solution is

$$y(x) = c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x}$$
$$= e^{-2x}(c_1 e^{ix} + c_2 e^{-ix})$$

The equivalent trigonometric form is

$$y(x) = e^{-2x}(a\cos x + b\sin x)$$

12. $y'' + 3y' + 5y = 0$

The characteristic equation

$$\lambda^2 + 3\lambda + 5 = 0$$

has complex roots

$$\lambda = \frac{1}{2}\left(-3 \pm \sqrt{9 - 20}\right) = -\frac{3}{2} \pm \frac{\sqrt{11}}{2} i$$

Then $y(x) = e^{-3x/2}\left[ae^{i\sqrt{11}x/2} + be^{-i\sqrt{11}x/2}\right]$

$$= e^{-3x/2}\left[A\cos\sqrt{11}x/2 + B\sin\sqrt{11}x/2\right]$$

Section 12.4

Solve the initial value problems:

13. $\dfrac{d^2x}{dt^2}+\dfrac{dx}{dt}-2x=0; \quad x(0)=1, \quad \dfrac{dx}{dt}(0)=0$

The characteristic equation is $\lambda^2+\lambda-2=(\lambda-1)(\lambda+2)=0$, with roots $\lambda=1,-2$.

The general solution is

$$x(t)=ae^{t}+be^{-2t}$$

with first derivative

$$\frac{dx}{dt}(t)=ae^{t}-2be^{-2t}$$

Then, by the initial conditions,

$$\left.\begin{aligned} x(0)&=1=a+b \\ \frac{dx}{dt}(0)&=0=a-2b \end{aligned}\right\} \;\rightarrow\; a=\frac{2}{3},\; b=\frac{1}{3}$$

The solution of the initial value problem is therefore

$$x(t)=\frac{1}{3}\left[2e^{t}+e^{-2t}\right]$$

14. $\dfrac{d^2x}{dt^2}+6\dfrac{dx}{dt}+9x=0; \quad x(1)=0, \quad \dfrac{dx}{dt}(1)=1$

The characteristic equation is $\lambda^2+6\lambda+9=(\lambda+3)^2=0$, with double root $\lambda=-3$.

The general solution is

$$x(t)=(a+bt)e^{-3t}$$

with first derivative

$$\frac{dx}{dt}(t)=(b-3a-3bt)e^{-3t}$$

Then, by the initial conditions,

$$\left.\begin{aligned} x(1)&=0=(a+b)e^{-3} \\ \frac{dx}{dt}(1)&=1=(-3a-2b)e^{-3} \end{aligned}\right\} \;\rightarrow\; a=-e^{3},\; b=e^{3}$$

and $\qquad x(t)=e^{3(1-t)}(t-1)$

15. $\dfrac{d^2x}{dt^2} + 9x = 0;\quad x(\pi/3) = 0,\ \dfrac{dx}{dt}(\pi/3) = -1$

The characteristic equation is $\lambda^2 + 9 = 0$, with complex roots $\lambda = \pm 3i$.

The general solution is, in trigonometric form,

$$x(t) = a\cos 3t + b\sin 3t$$

with first derivative

$$\frac{dx}{dt}(t) = -3a\sin 3t + 3b\cos 3t$$

Then, by the initial conditions,

$$\left.\begin{aligned} x(\pi/3) &= 0 = -a \\ \frac{dx}{dt}(\pi/3) &= -1 = -3b \end{aligned}\right\} \rightarrow \quad a = 0,\ b = \frac{1}{3}$$

and the solution of the initial-value problem is

$$x(t) = \frac{1}{3}\sin 3t$$

16. $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + 2x = 0;\quad x(0) = 1,\ \dfrac{dx}{dt}(0) = 0$

The characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$, with complex roots $\lambda = \frac{1}{2}(2 \pm \sqrt{4-8}) = 1 \pm i$.

The general solution is

$$x(t) = e^t(a\cos t + b\sin t)$$

with first derivative

$$\frac{dx}{dt}(t) = e^t\left[(a+b)\cos t + (b-a)\sin t\right]$$

Then, by the initial conditions,

$$\left.\begin{aligned} x(0) &= 1 = a \\ \frac{dx}{dt}(0) &= 0 = a + b \end{aligned}\right\} \rightarrow \quad a = 1,\ b = -1$$

and the solution of the initial-value problem is

$$x(t) = (\cos t - \sin t)e^t$$

Solve the boundary value problems:

17. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 8y = 0; \quad y(\pi/2) = -1, \;\; y(3\pi/4) = 1$

The characteristic equation is $\lambda^2 + 4\lambda + 8 = 0$, with complex roots $\lambda = \frac{1}{2}(-4 \pm \sqrt{16-32}) = -2 \pm 2i$.

The general solution is

$$y(x) = e^{-2x}(a\cos 2x + b\sin 2x)$$

Application of the boundary conditions gives

$$y(\pi/2) = -1 = e^{-\pi} \times (-a) \quad \rightarrow \quad a = e^{\pi}$$

$$y(3\pi/4) = 1 = e^{-3\pi/2} \times (-b) \rightarrow \quad b = -e^{3\pi/2}$$

and the solution of the boundary value problem is

$$y(x) = e^{\pi-2x}(\cos 2x - e^{\pi/2}\sin 2x)$$

18. $\dfrac{d^2y}{dx^2} + 9y = 0; \quad y = 0 \text{ when } x = 0, \;\; y = 1 \text{ when } x = \pi/2$

As in Exercise 15, the general solution of the differential equation is

$$y(x) = a\cos 3x + b\sin 3x$$

Then $\quad y(0) = 0 = a \quad \rightarrow \quad a = 0$

$$y(\pi/2) = 1 = -b \quad \rightarrow \quad b = -1$$

and the solution of the boundary value problem is

$$y(x) = -\sin 3x$$

19. $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0; \quad y(0) = 0, \;\; y(1) = 1$

The characteristic equation is $\lambda^2 + 8\lambda + 16 = 0$, with double root $\lambda = -4$.

The general solution is

$$y(x) = (a + bx)e^{-4x}$$

Then $\quad y(0) = 0 = a \quad \rightarrow \quad a = 0$

$$y(1) = 1 = be^{-4} \quad \rightarrow \quad b = e^{4}$$

Therefore $\quad y(x) = xe^{4(1-x)}$

20. $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$; $\quad y(0)=2,\ \ y\to 0$ as $x\to\infty$

As in Exercise 13, the general solution is

$$y(x)=ae^{x}+be^{-2x}$$

The first boundary condition gives

$$y(0)=2=a+b$$

The second condition requires that the solution go to zero as x goes to infinity. The function e^{-2x} has this property but the function e^{x} must be excluded. The condition therefore requires that we set $a=0$.

Then $b=2$ and the solution of the boundary value problem is

$$y(x)=2e^{-2x}$$

21. Solve $\frac{d^2\theta}{dt^2}+a^2\theta=0$ subject to the condition $\theta(t+2\pi\tau)=\theta(t)$.

The general solution of the differential equation is

$$\theta(t)=Ae^{iat}+Be^{-iat}.$$

Application of the cyclic boundary condition gives

$$\begin{aligned}\theta(t+2\pi\tau)&=Ae^{ia(t+2\pi\tau)}+Be^{-ia(t+2\pi\tau)}\\&=Ae^{iat}\times e^{i2\pi a\tau}+Be^{-iat}\times e^{-i2\pi a\tau}\\&=\theta(t)\ \text{ when }\ e^{\pm i2\pi a\tau}=1\end{aligned}$$

and the condition is satisfied when $2\pi a\tau=2\pi n$ for integer n. Therefore, $a=n/\tau$ and

$$\theta(t)=Ae^{int/\tau}+Be^{-int/\tau}, \qquad n=0,\pm1,\pm2,\cdots$$

Section 12.5

22. Given that the general solution of the equation of motion $m\ddot{x} = -kx$ for the harmonic oscillator is $x(t) = a\cos\omega t + b\sin\omega t$, where $\omega = \sqrt{k/m}$, **(i)** show that the solution can be written in the form $x(t) = A\cos(\omega t - \delta)$, where *A* is the amplitude of the vibration and δ is the phase angle, and express *A* and δ in terms of *a* and *b*; **(ii)** find the amplitude and phase angle for the initial conditions $x(0) = 1,\ \dot{x}(0) = \omega$.

(i) We have $x(t) = A\cos(\omega t - \delta) = A\cos\omega t\cos\delta + A\sin\omega t\sin\delta$

$$= a\cos\omega t + b\sin\omega t \quad \text{when} \quad \begin{cases} a = A\cos\delta \\ b = A\sin\delta \end{cases}$$

Therefore $A = \sqrt{a^2 + b^2}$, $\delta = \tan^{-1}(b/a)$

(ii) Application of the initial conditions gives

$$x(t) = a\cos\omega t + b\sin\omega t \quad \rightarrow \quad x(0) = 1 = a$$

$$\dot{x}(t) = -\omega a\sin\omega t + \omega b\cos\omega t \quad \rightarrow \quad \dot{x}(0) = \omega = \omega b$$

Therefore $a = b = 1 \ \rightarrow\ A = \sqrt{2},\ \ \delta = \tan^{-1}(1) = \dfrac{\pi}{4}$

and $x(t) = \sqrt{2}\,(\omega t - \pi/4)$

23. Solve the equation of motion for the harmonic oscillator with initial conditions $x(0) = 0,\ \dot{x}(0) = u_0$.

The general solution is

$$x(t) = a\cos\omega t + b\sin\omega t$$

Then $\dot{x}(t) = -\omega a\sin\omega t + \omega b\cos\omega t$

Therefore $\left.\begin{matrix} x(0) = 0 = a \\ \dot{x}(0) = u_0 = \omega b \end{matrix}\right\} \rightarrow \ x(t) = \dfrac{u_0}{\omega}\sin\omega t$

Section 12.6

24. For the particle in a box, find the nodes and sketch the graph of the wave function ψ_n for **(i)** $n = 4$ and **(ii)** $n = 5$.

By equation (12.53), the normalized wave functions are

$$\psi_n(x) = \sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l} \qquad n = 1, 2, 3, \ldots$$

(i) $n = 4$ $\quad \psi_4(x) = \sqrt{\frac{2}{l}} \sin \frac{4\pi x}{l}$

$$= 0 \text{ when } \frac{4\pi x}{l} = n\pi \quad \rightarrow \quad \frac{x}{l} = \frac{n}{4} = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$$

The sketch of $\psi_4(x)$ should look like Figure 1.

(ii) $n = 5$ $\quad \psi_5(x) = \sqrt{\frac{2}{l}} \sin \frac{5\pi x}{l}$

$$= 0 \text{ when } \frac{5\pi x}{l} = n\pi \quad \rightarrow \quad \frac{x}{l} = \frac{n}{5} = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$$

The sketch of $\psi_5(x)$ should look like Figure 2.

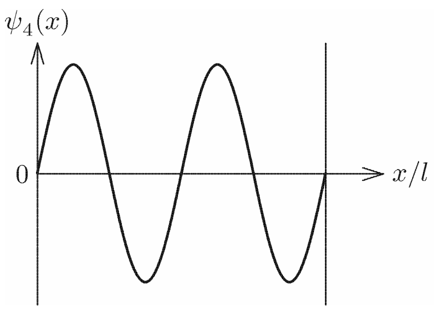


Figure 1

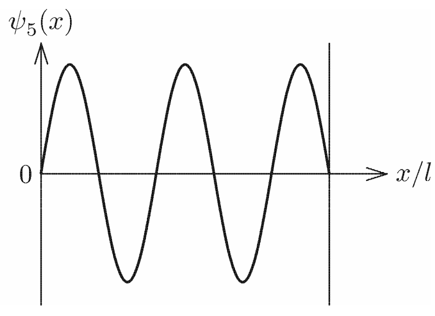


Figure 2

25. **(i)** Solve the Schrödinger equation (12.44) for the particle in a box of length l with potential-energy function $V = 0$ for $-l/2 \le x \le +l/2$, $V = \infty$ for $x \le -l/2$ and $x \ge +l/2$. **(ii)** Show that the solutions ψ_n are even functions of x when n is odd and odd functions when n is even. **(iii)** Show that the solutions are the same as those given by (12.53) if x is replaced by $x + l/2$, except for a possible change of sign.

(i) The general solution of the equation for the particle in a box of length l is, equation (12.48),

$$\psi(x) = d_1 \cos \omega x + d_2 \sin \omega x$$

where ω is related to the energy by $\omega^2 = 2mE/\hbar^2$.

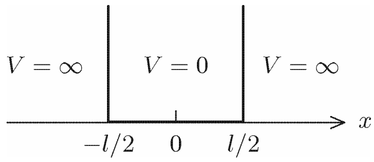


Figure 3

The boundary conditions in the present case are

$$\psi(-l/2) = \psi(+l/2) = 0$$

Then
$$\psi(-l/2) = 0 = d_1 \cos(-\omega l/2) + d_2 \sin(-\omega l/2)$$
$$= d_1 \cos(\omega l/2) - d_2 \sin(\omega l/2)$$
$$\psi(+l/2) = 0 = d_1 \cos(\omega l/2) + d_2 \sin(\omega l/2)$$

These conditions are satisfied in two ways:

(a) $d_2 = 0$ and $\cos(\omega l/2) = 0$

Then $\cos(\omega l/2) = 0$ when $\omega l/2 = n\pi/2$ for odd values of n,

with corresponding normalized wave functions (with $d_1 = \sqrt{2/l}$)

$$\psi_n(x) = \sqrt{\frac{2}{l}} \cos(n\pi x/l), \quad n = 1,\ 3,\ 5,\ \ldots$$

(b) $d_1 = 0$ and $\sin(\omega l/2) = 0$

$\sin(\omega l/2) = 0$ when $\omega l/2 = n\pi/2$ for even values of n,

with corresponding wave functions

$$\psi_n(x) = \sqrt{\frac{2}{l}} \sin(n\pi x/l), \quad n = 2,\ 4,\ 6,\ \ldots$$

(ii) (a) $\psi_n(x) = \sqrt{\frac{2}{l}} \cos(n\pi x/l) = +\sqrt{\frac{2}{l}} \cos(-n\pi x/l)$ is an even function of x

(b) $\psi_n(x) = \sqrt{\frac{2}{l}} \sin(n\pi x/l) = -\sqrt{\frac{2}{l}} \sin(-n\pi x/l)$ is an odd function of x

(iii) By equation (12.53),

$$\psi_n(x) = \sqrt{\frac{2}{l}}\sin(n\pi x/l), \quad n = 1,\ 2,\ 3,\ \ldots$$

Then $$\psi_n(x+l/2) = \sqrt{\frac{2}{l}}\sin\left(\frac{n\pi x}{l}+\frac{n\pi}{2}\right) = \sqrt{\frac{2}{l}}\left[\sin\frac{n\pi x}{l}\cos\frac{n\pi}{2}+\cos\frac{n\pi x}{l}\sin\frac{n\pi}{2}\right]$$

(a) n odd: $$\psi_n(x+l/2) = \sqrt{\frac{2}{l}}\left[\sin\frac{n\pi x}{l}\times 0+\cos\frac{n\pi x}{l}\times(\pm 1)\right] = (\pm)\sqrt{\frac{2}{l}}\cos\frac{n\pi x}{l}$$

(b) n even: $$\psi_n(x+l/2) = \sqrt{\frac{2}{l}}\left[\sin\frac{n\pi x}{l}\times(\pm 1)+\cos\frac{n\pi x}{l}\times 0\right] = (\pm)\sqrt{\frac{2}{l}}\sin\frac{n\pi x}{l}$$

26. For the particle in the box in Section 12.6, show that wave functions $\psi_n(x) = \sqrt{\frac{2}{l}}\sin\frac{n\pi x}{l}$ for $n = 1$ and $n = 2$ are **(i)** normalized, **(ii)** orthogonal.

(i) For normalization, $\int_0^l \psi_n(x)^2\,dx = \frac{2}{l}\int_0^l \sin^2\frac{n\pi x}{l}\,dx = 1$

We have $$\sin^2\frac{n\pi x}{l} = \frac{1}{2}\left[1-\cos\frac{2n\pi x}{l}\right]$$

Therefore $$\int_0^l \psi_n(x)^2\,dx = \frac{1}{l}\int_0^l\left[1-\cos\frac{2n\pi x}{l}\right]dx = \frac{1}{l}\left[x-\frac{l}{2n\pi}\sin\frac{2n\pi x}{l}\right]_0^l = \frac{1}{l}\big[(l)-(0)\big]$$

$$= 1 \text{ for all values of } n$$

(ii) For orthogonality, $\int_0^l \psi_1(x)\psi_2(x)\,dx = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\sin\frac{2\pi x}{l}\,dx = 0$

We have $$\sin ax\sin bx = \frac{1}{2}\big[\cos(a-b)x-\cos(a+b)x\big]$$

Therefore $$\int_0^l \psi_1(x)\psi_2(x)\,dx = \frac{1}{l}\int_0^l\left[\cos\frac{\pi x}{l}-\cos\frac{3\pi x}{l}\right]dx = \frac{1}{l}\left[\frac{l}{\pi}\sin\frac{\pi x}{l}-\frac{l}{3\pi}\sin\frac{3\pi x}{l}\right]_0^l$$

$$= 0$$

because $\sin n\pi x = 0$ for integer n.

Section 12.7

27. For the particle in a ring show that wave functions $\psi_n(\theta) = 1/\sqrt{2\pi}\, e^{in\theta}$ for $n = 3$ and $n = 4$ are **(i)** normalized, **(ii)** orthogonal.

(i) For normalization,
$$\int_0^{2\pi} \psi_n^*(\theta)\psi_n(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} \times e^{in\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta$$
$$= 1 \text{ for all values of } n$$

(ii) For orthogonality,
$$\int_0^{2\pi} \psi_3^*(\theta)\psi_4(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-3i\theta} \times e^{4i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta}\,d\theta$$
$$= \frac{1}{2\pi i}\Big[e^{i\theta}\Big]_0^{2\pi} = \frac{1}{2\pi i}\Big[e^{2\pi i} - e^0\Big] = 0$$

because for integer *n*.

28. The diagrams of Figure 12.8 are maps of the signs and nodes of some real wave functions (12.71) for the particle in a ring. Draw the corresponding diagrams for **(i)** $n = \pm 3$, **(ii)** $n = \pm 4$.

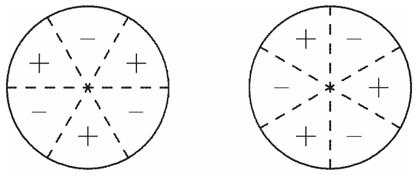
(i) $n = \pm 3$: we have (a) $\sin 3\theta = 0$ when $3\theta = n\pi,\ \theta = n\pi/3$

$$\rightarrow\ \theta = 0,\ \pi/3,\ 2\pi/3$$

(b) $\cos 3\theta = 0$ when $3\theta = n\pi/2,\ \theta = n\pi/6$ for n odd

$$\rightarrow\ \theta = \pi/6,\ \pi/2,\ 5\pi/6$$

Figure 4



$n = \pm 3$

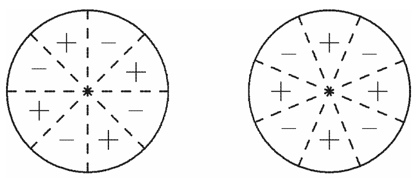
(i) $n = \pm 4$: (a) $\sin 4\theta = 0$ when $4\theta = n\pi,\ \theta = n\pi/4$

$$\rightarrow\ \theta = 0,\ \pi/4,\ \pi/2,\ 3\pi/4$$

(b) $\cos 4\theta = 0$ when $4\theta = n\pi/2,\ \theta = n\pi/8$ for n odd

$$\rightarrow\ \theta = \pi/8,\ 3\pi/8,\ 5\pi/8,\ 7\pi/8$$

Figure 5



$n = \pm 4$

29. Verify that equation (12.72) and its solutions (12.74) are transformed into (12.62) and (12.65) by means of the change of variable $\theta = x/r$.

Equation (12.72) is

$$\frac{d^2\psi}{dx^2} + \omega^2\psi = 0 \quad \text{where} \quad \omega^2 = 2mE/\hbar^2$$

Putting $x = r\theta$ (r constant),

$$\frac{d\psi}{dx} = \frac{d\psi}{d\theta}\frac{d\theta}{dx} = \frac{1}{r}\frac{d\psi}{d\theta}, \quad \frac{d^2\psi}{dx^2} = \frac{1}{r^2}\frac{d^2\psi}{d\theta^2}$$

Therefore $$\frac{d^2\psi}{dx^2} + \omega^2\psi = 0 \quad \to \quad \frac{d^2\psi}{d\theta^2} + r^2\omega^2\psi = 0$$

and $$r^2\omega^2 = \frac{2mr^2E}{\hbar^2} = \frac{2IE}{\hbar^2}$$

as required by equation (12.61). The solutions (12.74) are then

$$\psi_n = d_1 \cos\frac{2\pi nx}{l} + d_2 \sin\frac{2\pi nx}{l} \quad \to \quad d_1 \cos\frac{2\pi nr\theta}{l} + d_2 \sin\frac{2\pi nr\theta}{l}$$

and, because $2\pi r = l$,

$$\psi_n = d_1 \cos n\theta + d_2 \sin n\theta$$

and this is converted to the exponential form (12.65) by means of Euler's relations (8.35) and (8.36).

Section 12.8

30. Find a particular solution of the differential equation $y'' - y' - 6y = 2 + 3x$.

Let $y = a_0 + a_1 x$

Then $y' = a_1, \quad y'' = 0$

and $$y'' - y' - 6y = 2 + 3x \quad \to \quad -a_1 - 6a_0 - 6a_1 x = 2 + 3x$$
$$\to \quad a_0 = -1/4, \quad a_1 = -1/2$$

Therefore $$y = -\frac{1}{4} - \frac{x}{2}$$

Find the general solutions of the differential equations:

31. $y'' - y' - 6y = 2 + 3x$

By Exercise 7, the general solution of the homogeneous differential equation is $y_h = ae^{3x} + be^{-2x}$, and by Exercise 30, the particular integral is

$$y_p = -\frac{1}{4} - \frac{x}{2}$$

The general solution of the inhomogeneous equation is then

$$y = y_h + y_p = ae^{3x} + be^{-2x} - \frac{1}{4} - \frac{x}{2}$$

32. $y'' - 8y' + 16y = 1 - 4x^3$

By Exercise 9, the complementary function is $y_h = (a + bx)e^{4x}$. For the particular integral, let

$$y_p = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Then $\quad y_p' = a_1 + 2a_2 x + 3a_3 x^2, \quad y_p'' = 2a_2 + 6a_3 x$

and $\quad y_p'' - 8y_p' + 16y_p = (2a_2 - 8a_1 + 16a_0) + (6a_3 - 16a_2 + 16a_1)x + (-24a_3 + 16a_2)x^2 + 16a_3 x^3$

$$= 1 - 4x^3 \text{ when } a_3 = -\frac{1}{4},\ a_2 = -\frac{3}{8},\ a_1 = -\frac{9}{32},\ a_0 = -\frac{1}{32}$$

Therefore $\quad y_p = -\frac{1}{32} - \frac{9}{32}x - \frac{3}{8}x^2 - \frac{1}{4}x^3$

and $\quad y(x) = (a + bx)e^{4x} - \frac{1}{32}(1 + 9x + 12x^2 + 8x^3)$

33. $y'' - y' - 6y = 2e^{-3x}$

By Exercise 7, the complementary function is $y_h = ae^{3x} + be^{-2x}$

For the particular integral, let $y_p = ae^{-3x}$

Then $\quad y_p' = -3y_p, \ y_p'' = 9y_p$

and $\quad y_p'' - y_p' - 6y_p = (9 + 3 - 6)ae^{-3x} = 2e^{-3x}$ when $a = 1/3$

Therefore $\quad y(x) = ae^{3x} + be^{-2x} + \frac{1}{3}e^{-3x}$

34. $y'' - y' - 2y = 3e^{-x}$

The characteristic equation for the complementary function is

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0 \text{ when } \lambda = 2 \text{ and } \lambda = -1$$

and $y_h = ae^{2x} + be^{-x}$

By Table 12.1, case 1, the choice of particular integral should be $y_p = ke^{-x}$, but this is already a solution of the homogeneous equation. By prescription (a) therefore, we use

$$y_p = kxe^{-x}$$

Then $y'_p = k(1-x)e^{-x}$, $y''_p = k(-2+x)e^{-x}$

and $y''_p - y'_p - 2y_p = k(-2 + \cancel{x} - 1 + \cancel{x} - \cancel{2x})e^{-x}$

$$= -3ke^{-x} = 3e^{-x} \text{ when } k = -1$$

Therefore $y_p = -xe^{-x}$

and $y(x) = y_h + y_p = ae^{2x} + be^{-x} - xe^{-x} = ae^{2x} + (b-x)e^{-x}$

35. $y'' - 8y' + 16y = e^{4x}$

By Exercise 9, the complementary function is $y_h = (a + bx)e^{4x}$. By Table 12.1, case 1, the choice of particular integral should be $y_p = ke^{4x}$, but the characteristic equation for y_h has double root $\lambda = 4$.

By prescription (b) therefore, we use

$$y_p = kx^2e^{4x}$$

Then $y'_p = k(2x + 4x^2)e^{4x}$, $y''_p = k(2 + 16x + 16x^2)e^{4x}$

and $y''_p - 8y'_p + 16y_p = k(2 + \cancel{16x} + \cancel{16x^2} - \cancel{16x} - \cancel{32x^2} + \cancel{16x^2})e^{4x}$

$$= 2ke^{4x} = e^{4x} \text{ when } k = 1/2$$

Therefore $y_p = \frac{1}{2}x^2e^{4x}$

and $y(x) = y_h + y_p = (a + bx + x^2/2)e^{4x}$

36. $y'' - y' - 6y = 2\cos 3x$

By Exercise 7, the complementary function is $y_h = ae^{3x} + be^{-2x}$. For the particular integral, let

$$y_p = c\cos 3x + d\sin 3x$$

Then $\quad y_p' = -3c\sin 3x + 3d\cos 3x, \quad y_p'' = -9c\cos 3x - 9d\sin 3x$

and $\quad y_p'' - y_p' - 6y_p = (-15c - 3d)\cos 3x + (3c - 15d)\sin 3x$

$$= 2\cos 3x \text{ if } \begin{cases} 3c - 15d = 0 \rightarrow c = 5d \\ -15c - 3d = 2 \rightarrow d = -1/39,\ c = -5/39 \end{cases}$$

Therefore $\quad y_p = -\dfrac{1}{39}(5\cos 3x + \sin 3x)$

and $\quad y(x) = ae^{3x} + be^{-2x} - \dfrac{1}{39}(5\cos 3x + \sin 3x)$

37. $y'' + 4y = 3\sin 2x$

For the complementary function,

$$\lambda^2 + 4 = (\lambda + 2i)(\lambda - 2i) = 0 \text{ when } \lambda = \pm 2i$$

and $\quad y_h = a\cos 2x + b\sin 2x$

By Table 12.1, case 3, the choice of particular integral should be a combination of $\cos 2x$ and $\sin 2x$, but these are already solutions of the homogeneous equation. By prescription (a) therefore, we use

$$y_p = Cx\cos 2x + Dx\sin 2x$$

Then $\quad y_p' = (C + 2Dx)\cos 2x + (D - 2Cx)\sin 2x, \quad y_p'' = 4(D - Cx)\cos 2x - 4(C + Dx)\sin 2x$

and $\quad y_p'' + 4y = 4D\cos 2x - 4C\sin 2x$

$$= 3\sin 2x \text{ when } C = -3/4 \text{ and } D = 0$$

Therefore $\quad y_p = -\dfrac{3}{4}x\cos 2x$

and $\quad y(x) = y_h + y_p = (a - 3x/4)\cos 2x + b\sin 2x$

38. $y'' - y' - 6y = 2 + 3x + 2e^{-3x} + 2\cos 3x$

By Exercises 31, 33, and 36

$$y = ae^{3x} + be^{-2x} - \frac{1}{4} - \frac{x}{2} + \frac{1}{3}e^{-3x} - \frac{1}{39}(5\cos 3x + \sin 3x)$$

39. An RLC–circuit contains a resistor (resistance *R*), an inductor (inductance *L*), and a capacitor (capacitance *C*) connected in series with a source of e.m.f. *E*.

(i) Use Kirchhoff's voltage law (Section 11.7) to show that the current $I(t)$ in the circuit is given by the inhomogeneous equation

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{I}{C}=\frac{dE}{dt}$$

(ii) Find the solution of the homogeneous equation (for $dE/dt=0$), and confirm that it decays exponentially as $t\to\infty$.

(iii) Show that the particular integral for the periodic e.m.f. $E(t)=E_0\sin\omega t$ is

$$I_p(t)=I_0\sin(\omega t-\delta)$$

where $I_0=\dfrac{E_0}{\sqrt{R^2+S^2}}$, $\tan\delta=\dfrac{S}{R}$, and $S=\omega L-\dfrac{1}{\omega C}$.

(i) By Kirchhoff's law (Section 11.7),

$$E=E_L+E_R+E_C=L\frac{dI}{dt}+RI+\frac{Q}{C}$$

where $Q=\displaystyle\int I\,dt,\quad \frac{dQ}{dt}=I$

Then $\dfrac{dE}{dt}=L\dfrac{d^2I}{dt^2}+R\dfrac{dI}{dt}+\dfrac{I}{C}$

(ii) We have a second-order inhomogeneous differential equation. For the homogeneous equation

$$L\frac{d^2I_h}{dt^2}+R\frac{dI_h}{dt}+\frac{I_h}{C}=0$$

let $\alpha=R/2L,\ \beta^2=(R^2/4L^2)-(1/LC)=\alpha^2-(1/LC)$.

Then $\dfrac{d^2I_h}{dt^2}+2\alpha\dfrac{dI_h}{dt}+(\alpha^2-\beta^2)I_h=0$

whose characteristic roots are $\dfrac{1}{2}\left[-2\alpha\pm\sqrt{4\alpha^2-4(\alpha^2-\beta^2)}\right]=-\alpha\pm\beta$

Therefore $I_h=ae^{-(\alpha+\beta)t}+be^{-(\alpha-\beta)t}$

We consider two possible types of solution:

(a) β is real, so that $|\beta|<\alpha$. Then $\alpha\pm\beta>0$ and $I_h=ae^{-(\alpha+\beta)t}+be^{-(\alpha-\beta)t}\to 0$ as $t\to\infty$

(b) $\beta^2<0$, so that β is imaginary. Then $I_h=e^{-\alpha t}\left[A\cos\beta t+B\sin\beta t\right]\to 0$ as $t\to\infty$

(iii) If $E(t) = E_0 \sin \omega t$ then $\dfrac{dE}{dt} = E_0 \omega \cos \omega t$, and

$$L\frac{d^2 I_p}{dt^2} + R\frac{dI_p}{dt} + \frac{I_p}{C} = E_0 \omega \cos \omega t$$

Let $\quad I_p = a \cos \omega t + b \sin \omega t$

$$\frac{dI_p}{dt} = -a\omega \sin \omega t + b\omega \cos \omega t, \quad \frac{d^2 I_p}{dt^2} = -a\omega^2 \cos \omega t + b\omega^2 \sin \omega t$$

Then, if $S = \omega L - \dfrac{1}{\omega C}$,

$$\begin{aligned} L\frac{d^2 I_p}{dt^2} + R\frac{dI_p}{dt} + \frac{I_p}{C} &= \left[-a\omega^2 L + bR\omega + a/C\right]\cos \omega t + \left[-b\omega^2 L - aR\omega + b/C\right]\sin \omega t \\ &= \omega\left[bR - aS\right]\cos \omega t - \omega\left[bS + aR\right]\sin \omega t \\ &= E_0 \omega \cos \omega t \ \text{ when } \begin{cases} bR - aS = E_0 & \text{and} \\ bS + aR = 0 \end{cases} \end{aligned}$$

so that $\quad a = -\dfrac{SE_0}{R^2 + S^2}, \quad b = \dfrac{RE_0}{R^2 + S^2}$

Let $\quad \sin \delta = \dfrac{S}{\sqrt{R^2 + S^2}}, \quad \cos \delta = \dfrac{R}{\sqrt{R^2 + S^2}}$, and $I_0 = \dfrac{E_0}{\sqrt{R^2 + S^2}}$

Then $\quad a = -I_0 \sin \delta, \quad b = I_0 \cos \delta$

and $\quad I_p = -I_0 \sin \delta \cos \omega t \sin \delta + I_0 \cos \delta \sin \omega t \cos \delta$

$\quad = I_0 \sin(\omega t - \delta)$