The Chemistry Maths Book

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Solutions

Chapter 15 Orthogonal expansions. Fourier analysis

Section 15.2

1. Given the power series $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, use equation (15.9) to find the coefficient c_1 of $P_1(x)$ in the expansion (15.3) of $f(x)$ in Legendre polynomials.

We have $$c_1 = \frac{3}{2}\int_{-1}^{+1} P_1(x)\, f(x)\, dx = \frac{3}{2}\int_{-1}^{+1} x \sum_{l=0}^{\infty} a_l x^l\, dx = \frac{3}{2}\sum_{l=0}^{\infty} a_l \int_{-1}^{+1} x^{l+1}\, dx$$

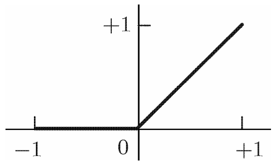
Now $$\int_{-1}^{+1} x^{l+1}\, dx = \left[\frac{x^{l+2}}{l+2}\right]_{-1}^{+1} = \begin{cases} 0 & \text{if } l \text{ is even} \\ \dfrac{2}{l+2} & \text{if } l \text{ is odd} \end{cases}$$

Therefore $$c_1 = 3\left[\frac{a_1}{3} + \frac{a_3}{5} + \frac{a_5}{7} + \cdots\right] = 3\sum_{l=0}^{\infty} \frac{a_{2l+1}}{2l+3}$$

2. (i) Find the first three terms of the expansion of the function

$$f(x) = \begin{cases} 0 & \text{if } \quad -1 < x < 0 \\ x & \text{if } \quad 0 < x < 1 \end{cases}$$

Figure 1



(Figure 1) in Legendre polynomials.

(ii) Sketch graphs of the one-term, two-term, and three-term representations of $f(x)$.

(i) Let $f(x) = \sum_{l=0}^{\infty} c_l P_l(x)$. Then

$$c_l = \frac{2l+1}{2}\int_{-1}^{+1} P_l(x)\, f(x)\, dx = \frac{2l+1}{2}\int_{0}^{+1} P_l(x)\, x\, dx$$

Using the Legendre polynomials listed in equations (15.4), we have

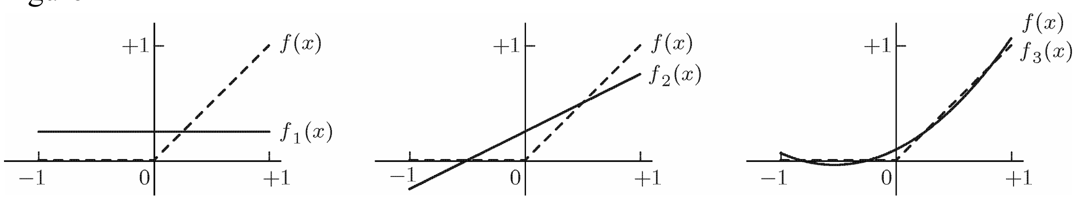
$$c_0 = \frac{1}{2}\int_0^{+1} x\, dx = \frac{1}{4}, \qquad c_1 = \frac{3}{2}\int_0^{+1} x^2\, dx = \frac{1}{2}, \qquad c_2 = \frac{5}{2}\int_0^{+1} \frac{1}{2}(3x^2 - 1)x\, dx = \frac{5}{16}$$

Therefore

$$f(x) = \frac{1}{4}P_0 + \frac{1}{2}P_1 + \frac{5}{16}P_2 + \cdots$$

(ii) $$f_1(x) = \frac{1}{4}P_0 = \frac{1}{4}; \quad f_2(x) = \frac{1}{4}P_0 + \frac{1}{2}P_1 = \frac{1}{4} + \frac{x}{2}; \quad f_3(x) = \frac{1}{4}P_0 + \frac{1}{2}P_1 + \frac{5}{16}P_2 = \frac{3}{32} + \frac{x}{2} + \frac{15}{32}x^2$$

Figure 2



Section 15.3

3. Show that the coefficient of $P_1(x)$ in the expansion $e^{itx} = \sum_{l=0}^{\infty} c_l P_l(x)$ is $c_1 = 3i\,j_1(t)$ where $j_1(t) = \dfrac{1}{t}\left(\dfrac{\sin t}{t} - \cos t\right)$.

If $$e^{itx} = \sum_{l=0}^{\infty} c_l P_l(x)$$

then $$c_l = \frac{2l+1}{2}\int_{-1}^{+1} P_l(x)e^{itx}\,dx$$

and $$c_1 = \frac{3}{2}\int_{-1}^{+1} xe^{itx}\,dx = \frac{3}{2}\left\{\left[\frac{xe^{itx}}{it}\right]_{-1}^{+1} - \frac{1}{it}\int_{-1}^{+1} e^{itx}\,dx\right\} = \frac{3}{2}\left\{\left[\frac{xe^{itx}}{it}\right]_{-1}^{+1} - \left[\frac{e^{itx}}{i^2t^2}\right]_{-1}^{+1}\right\}$$

$$= \frac{3}{2}\left\{\frac{1}{it}\left[e^{it} + e^{-it}\right] + \frac{1}{t^2}\left[e^{it} - e^{-it}\right]\right\}$$

$$= 3i\left\{\frac{1}{t^2}\sin t - \frac{1}{t}\cos t\right\} = 3i\,j_1(t)$$

where, by Example 13.12 and equations (13.59), $j_1(t)$ is the spherical Bessel function of order 1

4. Expand in terms of Legendre polynomials **(i)** $\cos tx$, **(ii)** $\sin tx$.

By equation (15.23),

$$e^{itx} = \cos tx + i\sin tx = \sum_{l=0}^{\infty}(2l+1)i^l j_l(t)P_l(x)$$

$$= \sum_{\substack{l=0\\(\text{even})}}^{\infty}(2l+1)i^l j_l(t)P_l(x) + \sum_{\substack{l=1\\(\text{odd})}}^{\infty}(2l+1)i^l j_l(t)P_l(x)$$

Therefore

(i) $$\cos tx = \sum_{\substack{l=0\\(\text{even})}}^{\infty}(2l+1)i^l j_l(t)P_l(x) = j_0(t)P_0(x) - 5j_2(t)P_2(x) + 9j_4(t)P_4(x) - \cdots$$

(ii) $$\sin tx = \frac{1}{i}\sum_{\substack{l=1\\(\text{odd})}}^{\infty}(2l+1)i^l j_l(t)P_l(x) = 3j_1(t)P_1(x) - 7j_3(t)P_3(x) + 11j_5(t)P_5(x) - \cdots$$

5. Find the first nonzero term in the expansion in powers of $1/R$ of the potential at point P for the system of three charges shown in Figure 3.

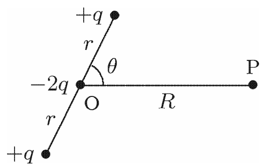


Figure 3

By equations (15.28) to (15.30),

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{Q_l}{R^{l+1}}$$

where, for the system of three charges,

$$Q_0 = \sum_{i=1}^{N} q_i = q - 2q + q = 0$$

$$Q_1 = \sum_{i=1}^{N} q_i r_i \cos\theta_i = qr\cos\theta - 2q\times 0 + qr\cos(\pi+\theta) = 0$$

$$Q_2 = \sum_{i=1}^{N} q_i r_i^2 \frac{1}{2}(3\cos^2\theta_i - 1) = qr^2\times\frac{1}{2}(3\cos^2\theta - 1) - 2q\times 0 + qr^2\times\frac{1}{2}(3\cos^2(\pi+\theta)-1)$$

$$= qr^2(3\cos^2\theta - 1)$$

Therefore $V = \dfrac{Q_2}{4\pi\varepsilon_0 R^3} + \cdots = \dfrac{qr^2}{4\pi\varepsilon_0 R^3}(3\cos^2\theta - 1) + \cdots$

6. For the square system of five charges in Figure 4, show that the leading term in the expansion in powers of $1/R$ of the electrostatic potential at P (in the plane of the square) is $V = qr^2/4\pi\varepsilon_0 R^3$, independent of orientation.

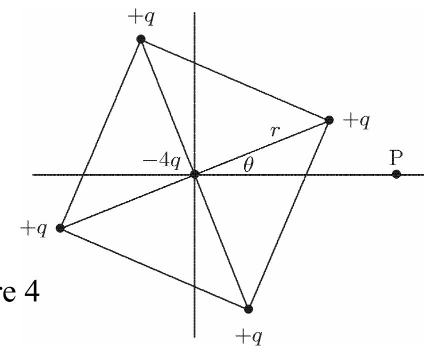


Figure 4

The system of 5 charges in Figure 15.18 can be considered as a combination of two systems of three charges of the type considered in Exercise 5, Figure 15.17, with orientations θ and $\theta + \pi/2$.

Then $Q_0 = 0$

$$Q_1 = 0$$

$$Q_2 = qr^2(3\cos^2\theta - 1) + qr^2(3\cos^2(\theta+\pi/2) - 1)$$

$$= qr^2\left[3(\cos^2\theta + \sin^2\theta) - 2\right] = qr^2 \qquad (\cos(\theta+\pi/2) = \sin\theta)$$

Therefore $V = \dfrac{qr^2}{4\pi\varepsilon_0 R^3} + \cdots$

Section 15.4

7. Confirm the relations **(i)** (15.33), **(ii)** (15.34) and **(iii)** (15.36).

(i) $\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = 0, \quad m \neq n$

We have $\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x)\right]$

Therefore $$\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = \frac{1}{2}\int_{-\pi}^{+\pi}\left[\cos(m-n)x - \cos(m+n)x)\right]dx$$

$$= \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{+\pi} \quad \text{(integral 4 in Table 6.1)}$$

But $\sin px = 0$ when p is an integer.

Therefore $$\int_{-\pi}^{+\pi} \sin mx \sin nx \, dx = 0$$

(ii) $\int_{-\pi}^{+\pi} \cos mx \sin nx \, dx = 0, \quad \text{all } m, n$

$\cos mx$ is an even function of x, $\sin nx$ is an odd function of x. The product $\cos mx \sin nx$ is therefore odd, and the integral is zero.

(iii) $\int_{-\pi}^{+\pi} \sin nx \sin nx \, dx = \pi \quad \text{if } n > 0$

We have $\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

Therefore $$\int_{-\pi}^{+\pi} \sin^2 nx \, dx = \frac{1}{2}\int_{-\pi}^{+\pi}(1 - \cos 2nx)\, dx$$

$$= \left[x - \frac{1}{2n}\sin 2nx\right]_{-\pi}^{+\pi} = \pi \quad \text{(integral 2 in Table 6.1)}$$

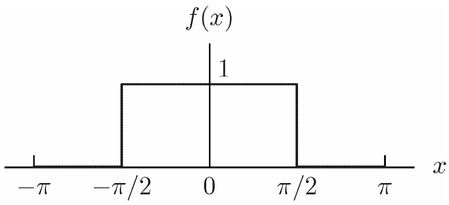
8. A periodic function with period 2π is defined by

$$f(x)=\begin{cases}1 & \text{if } -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}\\[2ex] 0 & \text{if } \dfrac{\pi}{2}<|x|<\pi\end{cases}$$

(i) Draw the graph of the function in the interval $-3\pi \le x \le 3\pi$. **(ii)** Find the Fourier series of the function [Hint: $f(x)$ is an even function of *x*]. **(iii)** Use the series to show that

$\dfrac{\pi}{4}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots$ [Hint: substitute a suitable value for *x* in the series].

(i) Figure 5



(ii) The Fourier series (15.37) is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

In the present case, $f(x)$ is an even function in the interval $-\pi<x<\pi$ and only the even trigonometric functions $\cos nx$ contribute (all $b_n=0$):

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \qquad \text{(Fourier cosine series)}$$

We have $\quad a_0=\dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}dx=1$

and $\quad a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi}f(x)\cos nx\,dx=\dfrac{2}{\pi}\int_0^{\pi/2}\cos nx\,dx$

$$=\frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi/2}=\begin{cases}0 & \text{if } n \text{ even}\\ 2/n\pi & \text{if } n=1,5,9\ldots\\ -2/n\pi & \text{if } n=3,7,11\ldots\end{cases}$$

Then $\quad f(x)=\dfrac{1}{2}+\dfrac{2}{\pi}\left[\dfrac{\cos x}{1}-\dfrac{\cos 3x}{3}+\dfrac{\cos 5x}{5}-\dfrac{\cos 7x}{7}+\cdots\right]$

(iii) Put $x=0$

Then $\quad f(0)=1=\dfrac{1}{2}+\dfrac{2}{\pi}\left[1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots\right]$

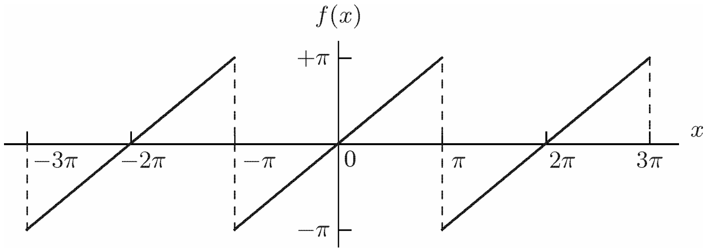
and $\quad \dfrac{\pi}{4}=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\cdots$

9. A function with period 2π is defined by

$$f(x) = x, \qquad -\pi < x < \pi$$

(i) Draw the graph of the function in the interval $-3\pi \le x \le 3\pi$. **(ii)** Find the Fourier series of the function. [Hint: $f(x)$ is an odd function of x] **(iii)** Draw the graphs of the first four partial sums of the series.

(i) Figure 6



(ii) Function $f(x)$ is an odd function in the interval $-\pi < x < \pi$ and only the odd trigonometric functions $\sin nx$ contribute to the Fourier series (all $a_n = 0$):

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{(Fourier sine series)}$$

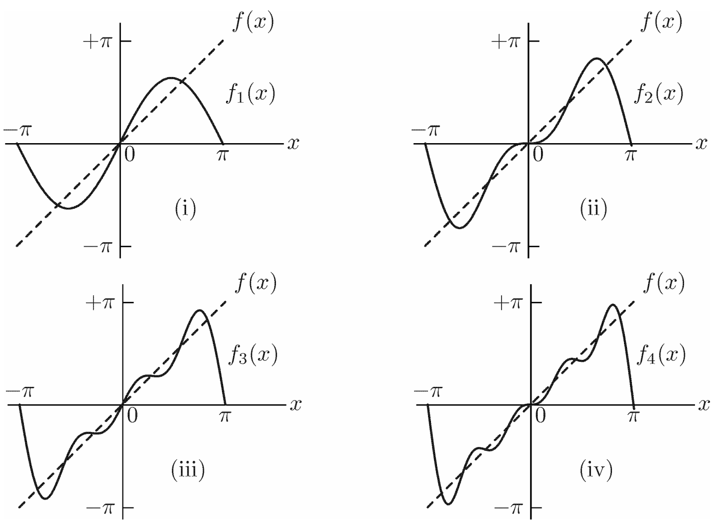
We have $\quad b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} f(x)\sin nx\,dx = \dfrac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$

$$= \frac{2}{\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{+\pi}\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right\} = -\frac{2}{n}\cos n\pi \quad \begin{cases} +2/n & \text{if } n \text{ odd} \\ -2/n & \text{if } n \text{ even} \end{cases}$$

Then $\quad f(x) = 2\left[\dfrac{\sin x}{1} - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \dfrac{\sin 4x}{4} + \cdots\right]$

(iii) Figure 7



10. A function with period 2π is defined by

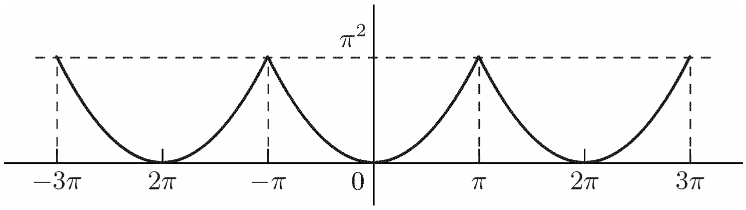
$$f(x) = x^2, \qquad -\pi \le x \le \pi$$

(i) Draw the graph of the function in the interval $-3\pi \le x \le 3\pi$. **(ii)** Find the Fourier series of the function. **(iii)** Use the series to show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

(i) Figure 8



(ii) Function $f(x)$ is an even function in the interval $-\pi < x < \pi$ and only the even trigonometric functions $\cos nx$ contribute:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

and, by parts (as in Example 6.11),

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx = \frac{4}{n^2}\cos n\pi = \begin{cases} -4/n^2 & \text{if } n \text{ odd} \\ +4/n^2 & \text{if } n \text{ even} \end{cases}$$

Therefore $$f(x) = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\right]$$

(iii) Put $x = \pi$: $$f(\pi) = \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

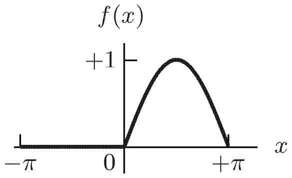
$$\rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Put $x = 0$: $$f(0) = 0 = \frac{\pi^2}{3} - 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\rightarrow \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

11. Show that the Fourier series of the periodic function defined by (see Figure 15.4)

$$f(t)=\begin{cases}\sin t & \text{if}\quad 0\le t\le \pi\\ 0 & \text{if}\ -\pi\le t\le 0\end{cases}$$



is $\dfrac{1}{\pi}+\dfrac{1}{2}\sin t-\dfrac{2}{\pi}\left[\dfrac{\cos 2t}{1\cdot 3}+\dfrac{\cos 4t}{3\cdot 5}+\dfrac{\cos 6t}{5\cdot 7}+\cdots\right]$

We have $\quad f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

(*a*) $n=0$: $\quad a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} f(x)\,dx=\dfrac{1}{\pi}\int_0^{\pi}\sin x\,dx=\dfrac{2}{\pi}$

$n=1$: $\quad a_1=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\sin x\cos x\,dx=\left[\dfrac{\sin^2 x}{2}\right]_0^{\pi}=0$

$n>1$: $\quad a_n=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\sin x\cos nx\,dx=\dfrac{1}{2\pi}\int_0^{\pi}\left[\sin(n+1)x-\sin(n-1)x\right]dx$

$$=\frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}=\frac{1}{2\pi}\left[\frac{-\cos(n+1)\pi+1}{n+1}+\frac{\cos(n-1)\pi-1}{n-1}\right]$$

$$=\begin{cases}0 & \text{if } n \text{ odd}\qquad [\cos(n\pm 1)\pi=1 \text{ if } n \text{ odd}]\\ -\dfrac{2}{(n-1)(n+1)\pi} & \text{if } n \text{ even}\end{cases}$$

(*b*) $n=1$: $\quad b_1=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\sin^2 x\,dx=\dfrac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\,dx=\dfrac{1}{2}$

$n>1$: $\quad b_n=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\sin x\sin nx\,dx=\dfrac{1}{2\pi}\int_0^{\pi}\left[\cos(n-1)x-\cos(n+1)x\right]dx$

$$=\frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1}-\frac{\sin(n+1)x}{n+1}\right]_0^{\pi}=0\qquad [\sin(n\pm 1)\pi=\sin 0=0]$$

Therefore

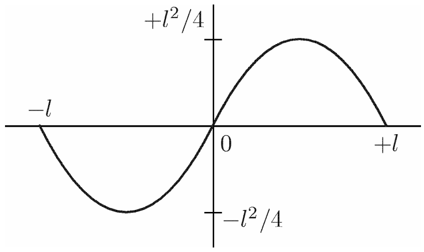
$$f(x)=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\left[\frac{1}{1\cdot 3}\cos 2x+\frac{1}{3\cdot 5}\cos 4x+\frac{1}{5\cdot 7}\cos 6x+\cdots\right]$$

12. A function with period $2l$ is defined by

$$f(x) = \begin{cases} x(l-x) & \text{if} \quad 0 < x < l \\ x(l+x) & \text{if} \quad -l < x < 0 \end{cases}$$

(i) Draw the graph of the function in the interval $-l \le x \le l$. **(ii)** Find the Fourier series of the function.

(i) Figure 9



(ii) Function $f(x)$ is an odd function in the interval $-l < x < l$ and only the odd trigonometric functions $\sin n\pi x/l$ contribute to the Fourier series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Then (integrating by parts),

$$b_n = \frac{1}{l}\int_{-l}^{+l} f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_{0}^{l} x(l-x) \sin \frac{n\pi x}{l}\, dx = \frac{4l^2}{n^3\pi^3}(1-\cos n\pi) = \begin{cases} 0 & \text{if } n \text{ even} \\ \dfrac{8l^2}{n^3\pi^3} & \text{if } n \text{ odd} \end{cases}$$

Therefore $$F(x) = \frac{8l^2}{\pi^3}\left[\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \cdots\right]$$

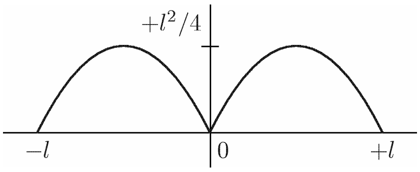
$$= \frac{8l^2}{\pi^3} \sum_{\substack{n=1 \\ (\text{odd})}}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l}$$

13. A function with period $2l$ is defined by

$$f(x) = \begin{cases} x(l-x) & \text{if} \quad 0 < x < l \\ -x(l+x) & \text{if} \quad -l < x < 0 \end{cases}$$

(i) Draw the graph of the function in the interval $-l \le x \le l$. **(ii)** Find the Fourier series of the function.

(i) Figure 10



(ii) Function $f(x)$ is an even function in the interval $-l < x < l$ and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Then $$a_0 = \frac{2}{l}\int_0^l x(l-x)\,dx = \frac{l^2}{3}$$

and (integrating by parts),

$$a_n == \frac{2}{l}\int_0^l x(l-x)\cos\frac{n\pi x}{l}\,dx$$

$$= -\frac{2l^2}{n^2\pi^2}(1+\cos n\pi) = \begin{cases} -\dfrac{4l^2}{n^2\pi^2} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

Therefore $$f(x) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{4^2}\cos\frac{4\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \cdots\right]$$

Section 15.5

14. Find the solution of the diffusion equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

in the interval $0 \le x \le l$ that satisfies the boundary conditions $T(0, t) = T(l, t) = 0$ and initial condition $T(x, 0) = x(l - x)$, and that decreases exponentially with time (See Exercise 17 of Chapter 14).

We consider the solution in 4 steps

(1) Separation of variables:
Put $T(x,t) = F(x)\times G(t)$, substitute in the diffusion equation, divide by $T = F\times G$. Then

$$\frac{1}{F}\frac{d^2F}{dx^2} = \frac{1}{DG}\frac{dG}{dt} = C \quad\to\quad \frac{d^2F}{dx^2} = CF, \quad \frac{dG}{dt} = CDG$$

(2) Solution of the equation in *x*:
The boundary conditions $T(0,t) = T(l,t) = 0$ mean, for *x*, that $F(0) = F(l) = 0$. These are the boundary conditions for the particle in a box discussed in Section 12.6, and of the vibrating string in Section 12.7. The allowed values of the separation constant are therefore $C_n = -n^2\pi^2/l^2$ and particular solutions are

$$F_n(x) = \sin\frac{n\pi x}{l}$$

(3) Solution of the equation in *t* for each value of *n*:

$$\frac{dG}{dt} = -\frac{n^2\pi^2 D}{l^2}G \quad\to\quad G_n(t) = e^{-n^2\pi^2 Dt/l^2}$$

(4) The particular solution for each *n* is $T_n(x,t) = F_n(x)G_n(t)$ and the general solution that satisfies the boundary conditions is

$$T(x,t) = \sum_{n=1}^{\infty} A_n F_n(x) G_n(t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}e^{-n^2\pi^2 Dt/l^2}$$

(5) The initial condition is $T(x, 0) = x(l - x)$. Then

$$T(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = x(l-x)$$

By Exercise 12 above, the Fourier series for $x(l-x)$ is $\frac{8l^2}{\pi^3}\sum_{\substack{n=1\\ \text{(odd)}}}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{l}$

Then $\quad T(x,0) = x(l-x) \quad$ when $\quad A_n = \begin{cases} 0 & \text{if } n \text{ even} \\ \dfrac{8l^2}{n^3\pi^3} & \text{if } n \text{ odd} \end{cases}$

and the particular solution of the diffusion equation that satisfies both boundary and initial conditions is

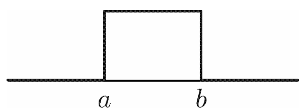
$$T(x,t) = \frac{8l^2}{\pi^3} \sum_{\substack{n=1 \\ (\text{odd})}}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l} e^{-n^2\pi^2 Dt/l^2}$$

Section 15.6

Find the Fourier transform:

15. $f(x) = \begin{cases} 1, & \text{if } \quad a < x < b \\ 0, & \text{otherwise} \end{cases}$

Figure 11



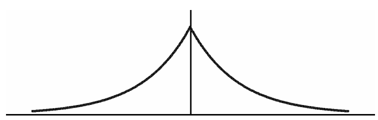
We have $\quad g(y) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} f(x) e^{-ixy}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-ixy}\, dx = \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{iy} e^{-ixy} \right]_{x=a}^{x=b} = \frac{e^{-iay} - e^{-iby}}{i\sqrt{2\pi}\, y}$$

16. $f(x) = e^{-a|x|} \quad (a > 0)$

We have $\quad e^{-a|x|} = \begin{cases} e^{-ax} & x > 0 \\ e^{ax} & x < 0 \end{cases}$

Figure 12



Therefore $\quad g(y) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} f(x) e^{-ixy}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\infty} e^{-ax} e^{-ixy}\, dx + \int_{-\infty}^{0} e^{ax} e^{-ixy}\, dx \right]$$

Replacement of x by $-x$ in the second integral gives

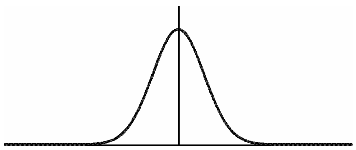
$$g(y) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[e^{-ax} e^{-ixy} + e^{-ax} e^{+ixy} \right] dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[e^{-(a+iy)x} + e^{-(a-iy)x} \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{e^{-(a+iy)x}}{a+iy} - \frac{e^{-(a-iy)x}}{a-iy} \right]_0^{\infty} = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+iy} + \frac{1}{a-iy} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + y^2} \right)$$

17. Show that the Fourier transform of e^{-ax^2} $(a > 0)$ is

$$\frac{1}{\sqrt{2a}} e^{-y^2/4a}$$ (see Figure 15.14b)



$$\left[\text{Hint : change variable to } t = \sqrt{a}\,x + \frac{iy}{2\sqrt{a}} \text{ and use } \int_{-\infty}^{\infty} e^{-\alpha t^2}\, dt = \sqrt{\frac{\pi}{\alpha}}\right]$$

We have $$g(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ax^2} e^{-ixy}\, dx$$

Let $$t = \sqrt{a}\,x + \frac{iy}{2\sqrt{a}} \quad \to \quad dt = \sqrt{a}\, dx\,.$$

Then $$t^2 = ax^2 + ixy - \frac{y^2}{4a}$$

$$e^{-t^2} = e^{y^2/4a} \times e^{-ax^2} e^{-ixy}$$

and $$g(y) = \frac{e^{-y^2/4a}}{\sqrt{2\pi a}} \int_{-\infty}^{+\infty} e^{-t^2}\, dt = \frac{e^{-y^2/4a}}{\sqrt{2\pi a}} \times \sqrt{\pi}$$

$$= \frac{e^{-y^2/4a}}{\sqrt{2a}}$$